AB EXAM

Texas A&M High School Math Contest

November 12, 2022

Directions: *All answers should be simplified and if units are involved include them in your answer.*

1. An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise?

   Solution. The brick has a volume of $40 \cdot 20 \cdot 10 = 8000$ cm^3. Suppose that after the brick is placed in the tank, the water level rises by h centimeters. Then the additional volume occupied in the aquarium is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters. Since this must be the same as the volume of the brick, we have $8000 = 4000h$ and $h = 2$ cm.

   Answer: 2 cm

2. A train moves at a constant speed. It takes 10 seconds for the entire train to pass a standing observer, and it takes 30 seconds for the entire train to completely cross a bridge 400 meters long. What is the length of the train (in meters)?

   Solution Let L be the length of the brige and v be the length of the train. Then from the condition that the train passes by a standing observer in 10 seconds it follows that $v = \frac{L}{10}$, while from the condition that it passes the bridge of legth 400 meters in 40 seconds it follows that $v = \dfrac{L+400}{30}$. Hence $\dfrac{L+400}{30} = \dfrac{L}{10} \Leftrightarrow L + 400 = 3L \Leftrightarrow L = 200$ m.
   Answer 200 m.

3. Eli, Joshua, and Luke are brothers. Eli is 2 years older than Joshua; and Joshua is 1 year less than three times as old as Luke. Together, they are 14 years old. How old is Eli?

   Solution Let E, J, and L are the ages of Eli, Joshua, and Luke, respectively.

   $E = J + 2$ and $J = 3L - 1 \Rightarrow E = 3L + 1$. Consequently, $14 = 3L + 1 + 3L - 1 + L = 7L \Rightarrow L = 2 \Rightarrow E = 3L + 1 = 7$. **Answer:** 7 years old.

4. The numerator and denominator of a fraction are positive integers summing up to 101 and the fraction is not greater then $\frac{1}{3}$. Find the largest possible value of the fraction.

   Solution

   Let numerator be equal to m , then the denominator is equal to $101 - m$. As numerator increases then the denominator decreases so as long the denominator is positive the fraction $\frac{m}{101-m}$ increases as m increases. The condition $\dfrac{m}{101-m} \leq \frac{1}{3}$ is equivalent to $4m \leq 101$ and the maximal integer m satisfying it is $m = 25$. Therefore the maximal fraction satisfying the desired property is $\frac{25}{101-25} = \frac{25}{76}$.

   Answer $\dfrac{25}{76}$.

5. Flickering Fred's Christmas lights aren't working. Initially, Fred has 90% of his light bulbs working. After changing some of them in the morning of one day, Fred has 94% of his light bulbs working. He

returned to work in the afternoon of the same day and changed 9 less than twice the number of bulbs he changed in the morning. After this Fred has 99% of his light bulbs working. If the only way to make a broken light work is to change the bulb, how many light bulbs does Fred have working when 99% of his lights are working?

Solution Let x be the number of bulbs Fred changed in the morning. Then he changed $2x-9$ bulbs in the afternoon. Then $\frac{2x-9}{x}=\frac{5}{4}$, which implies that $x=12$. Now 12 bulbs is 4% of the of the total, hence the total is 300. Hence, the number of the working Lights $= 0.99 \cdot 300 = 297$ working lights.

Answer: 297.

6. It is known that $\frac{x+y}{x-y}+\frac{x-y}{x+y}=3$. Find the value of $\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}$.

   Solution Observe that

$$3=\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{(x+y)^2+(x-y)^2}{(x-y)(x+y)}=\frac{x^2+\cancel{2xy}+y^2+x^2-\cancel{2xy}+y^2}{x^2-y^2}=2\frac{x^2+y^2}{x^2-y^2}.$$

   Hence

$$\frac{x^2+y^2}{x^2-y^2}=\frac{3}{2} \quad \Rightarrow \quad \frac{x^2-y^2}{x^2+y^2}=\frac{2}{3}.$$

   So,

$$\frac{x^2+y^2}{x^2-y^2}+\frac{x^2-y^2}{x^2+y^2}=\frac{3}{2}+\frac{2}{3}=\frac{9+4}{6}=\frac{13}{6}.$$

   Answer: $\frac{13}{6}$.

7. Each member of a family drank a full cup of coffee with milk over breakfast (milk is just a usual liquid milk). All cups are identical. It is known that Alice, a member of the family, drank 1/4 of the total amount of milk and 1/6 of the total amount of coffee consumed by the family during this meal. It is assumed that some (nonzero) amount of coffee and some (nonzero) amount of milk was consumed. How many members are in this family?

   Solution. Assume that the family has n members, c is the total amount of consumed coffee and m is the total amount of consumed milk . Then the volume of one cup can be represented in two ways: on one hand, counting the volume of Alice's cup we get $\frac{c}{6}+\frac{m}{4}$. On the other hand, since all cups are identical, it is equal to $\frac{1}{n}(c+m)$, hence

$$\frac{c}{6}+\frac{m}{4}=\frac{1}{n}(c+m) \Leftrightarrow \left(\frac{1}{n}-\frac{1}{6}\right)c=\left(\frac{1}{4}-\frac{1}{n}\right)m$$

   Since both c and m are positive, then $\frac{1}{n}-\frac{1}{6}$ and $\frac{1}{4}-\frac{1}{n}$ are of the same sign (and none of them is zero), This is possible if and only if $n=5$.

   Answer 5.

8. Frolicking Frank decides to take a trip across Texas. Frank starts in El Paso and travels 850 miles to Port Arthur. He starts to travel with certain constant speed. However, making exactly half of the distance Frank realizes that he is running late and decides to increase his speed by 18 miles per hour for the rest of his trip to Port Arthur. It is known that the total time he spent on this travel is the

same as he would travel all the way with constant velocity which is 8 miles per hour faster than his initial speed. What was his initial speed?

Solution. In fact, the distance from El Paso to Port Artur is not important. If we denote it by L it will be canceled in the equation. Assume that Frolicking Frank initial speed is x. Then the condition of the problem can be written as the following equation:

$$\frac{\cancel{L}}{2x} + \frac{\cancel{L}}{2(x+18)} = \frac{\cancel{L}}{x+8} \Leftrightarrow \frac{1}{2x} + \frac{1}{2(x+18)} = \frac{1}{x+8} \Leftrightarrow (4x+36)(x+8) = 4x(x+18) \Leftrightarrow$$

$$(x+9)(x+8) = x^2 + 18x \Leftrightarrow \cancel{x^2} + 17x + 72 = \cancel{x^2} + 18x \Leftrightarrow x = 72.$$

Answer: 72 mph

9. Find the largest x such that

$$\frac{4^{x^2+5}}{8^{4x}} = (0.125)^{3x-5}$$

Solution Replace all by powers of 2 using that $4 = 2^2$, $8 = 2^3$, and $0.125 = 2^{-3}$: $2^{2x^2+10-12x} = 2^{-3(3x-5)} \Leftrightarrow 2x^2 - 12x + 10 = -9x + 15 \Rightarrow 2x^2 - 3x - 5 = 0$. The discriminant $D = 3^2 - 4 \cdot 2 \cdot (-5) = 9 + 40 = 49$, $x_1 = \frac{3-7}{4} = -1$, $x_2 = \frac{3+7}{4} = \frac{5}{2} = 2.5$. So, $x_2 = \frac{5}{2} = 2.5$ is the largest solution

Answer $\frac{5}{2} = 2.5$

10. If dividing $x^3 - kx^2 + 5x + 8$ by $x - 1$ yields a remainder of 5, find k.

Solution Let $f(x) = x^3 - kx^2 + 5x + 8$. By Remainder Theorem, $f(1) = 5$. Thus, $1^3 - k(1)^2 + 5(1) + 8 = 5$ $\Rightarrow k = 9$

Answer: 9

11. There are two types of trees growing in the park, oaks and maples. At the beginning of this year oaks constituted 60% of all trees. New trees were planted two times during the year, in Spring and in Fall. In Spring only maples were planted and after this procedure the oaks constituted 20% of all trees at the park. In Fall only oaks were planted and after this procedure they again constituted 60% of all trees. Let e be the number of trees in the park at the end of the year and b be the number of trees in the park at the beginning of the year. Find the ratio $\frac{e}{b}$. It is assumed that not a single tree in the park was cut down or fell by itself during the year.

Solution Assume that the number of oaks in the beginning of the year is x, then the number of maples is $\frac{2}{3}x$. After the Spring planting the number of maples became 4 times the number of oaks, but the latter did not change, so the number of maples became $4x$. After the Fall planting the number of oaks becomes $\frac{3}{2}$ of the number of maples, but the latter did not change, so the number of oaks becomes $6x$. Consequently, $e = 10x$ and $b = (1 + \frac{2}{3})x = \frac{5}{3}x$, so the required ratio is $10 : \frac{5}{3} = 6$.

One can alternatively argue that the proportion of the number of oaks and maples after the Fall planting returns to the one at the beginning of the year, and the number of maples is still $4x$. Therefore the desired ratio is equal to the ratio of the maples at the end and at the beginning of the year, which is $4 : \frac{2}{3} = 6$.

Answer 6.

12. The sum of the squares of the roots of the polynomial $2x^2 + x + c$ is equal to 5. Find c.

Solutions If x_1 and x_2 are the roots of $2x^2 + x + c$, then by the Vieta theorem $x_1 + x_2 = -\frac{1}{2}$ and $x_1 x_2 = \frac{c}{2}$. Hence

$$5 = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2 = \frac{1}{4} - c \;\Rightarrow\; c = -\frac{19}{4}.$$

Answer: $-\frac{19}{4}$.

13. What is the sum of the digits in the number $10^{55} - 55$?

Solution. 10^{55} has 55 zero digits and

$$10^{55} - 55 = \underbrace{9 \ldots 9}_{53\ digits}45.$$

The sum of the digits is $53 \cdot 9 + 4 + 5 = 9 \cdot 54 = 486$.

Answer: 486

14. The digits of a three-digit number A are reversed and the resulting number is summed with the original number to get 1332. What is the smallest possible value for A?

Solution. Let $A = 100x + 10y + z$. Then by assumptions: $(100x + 10y + z) + (100z + 10y + x) = 100(x + z) + 10(2y) + (x + z) = 1332$. Observing that the tens digit $(2y)$ is an odd value, we know that $x + z \geq 10$. Thus, $x + z = 12$. Using similar logic, $2y > 10$, because we have 13 from the very left in the resulting sum but $x + z = 12$. So $2y = 12$ and therefore $y = 6$. If we want the smallest value of A, then from $x + z = 12$ we get $x = 3$ and $z = 9$. Thus, $A = 369$.

Answer 369.

15. Find a positive four-digit integer which has the decimal representation $(abba)_{10}$ and is a perfect cube.

Solution. Let $n = (abba)_{10}$. We have

$$(abba)_{10} = 1000a + 100b + 10b + a = 1001a + 110b = 11(91a + 10b).$$

Therefore n is divisible by 11. Since n is also a perfect cube, we conclude that $n = 11^3 k^3 = 1331k^3$ for some integer k. On the other hand $1331k^3 > 10000$ for $k > 1$. Thus, $k = 1$ and $n = 11^3 = 1331$.

Answer: 11^3 or 1331

16. Given p and q are positive integers satisfying the following equality $p! + 12 = q^2$, find $p + q$. (Here $p! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-1) \cdot p$, the *factorial of* p.)

Solution. First note that if $p \geq 5$ then $p!$ is proportional to 2 and to 5, i.e. $p!$ is divisible by 10 and thus the number $p! + 12$ has 2 as its last digit. But such a number cannot be a perfect square of an integer. It remains to consider cases $p = 1, 2, 3, 4$. It is easy to conclude that only the pair $(p, q) = (4, 6)$ satisfies the given equation. Finally, $p + q = 10$.

Answer: 10

17. If $f(1) = 5$ and $f(n+1) = 2f(n) + 1$ for all integers $n \geq 1$, find $f(11)$.

Solution Observe that $f(n+1)+1 = 2f(n)+2 = 2(f(n)+1)$. Therefore, $f(n)+1 = 2^{n-1}(f(1)+1) = 2^{n-1} \cdot 6 = 3 \cdot 2^n \Rightarrow f(n) = 3 \cdot 2^n - 1 \Rightarrow f(11) = 3 \cdot 2^{11} - 1 = 3 \cdot 2048 - 1 = 6143$

Answer: 6143.

18. Represent the number $(101011011011)_2 + (301021)_4 + (6711)_8$ in the base 8. Here $(x_1 x_2 \dots x_k)_b$ denotes the number having digits $x_1, \cdots, x_k$ in base b.

Solution Convert each summand into base 8. First, since $8 = 2^3$ to translate the representation of a number in base 2 to the representation in base 8 we have to subdivide the binary representation into triples starting from the right and convert each triple to the corresponding digit from 0 to 7. In this way we get:

$$(\underbrace{101}_{5}\underbrace{011}_{3}\underbrace{011}_{3}\underbrace{011}_{3})_2 = (5333)_8$$

Since $4 = 2^2$ then to translate the representation of a number in base 4 to the representation in base 2 one has to replace every its digit in base 4 to a pair of 0's and 1's corresponding to the representation of this digit in base 2. In this way, and using the same strategy as in the previous step we get:

$$301021_4 = (11\,00\,01\,00\,10\,01)_2 = (\underbrace{110}_{6}\underbrace{001}_{1}\underbrace{001}_{1}\underbrace{001}_{1})_2 = (6111)_8$$

Therefore the required representation is equal to:

$$(5333)_8 + (6111)_8 + (6711)_8 = (22355)_8$$

Answer $(22355)_8$

19. Twelve numbers are written along the circle. It is known that each number is equal to the absolute value of the differences of two numbers next to it in clockwise direction. It is also known that the sum of these numbers is equal to 1. What is the sum of their cubes?

Solution Since each of these numbers is equal to absolute values of some other number, they are nonnegative. Assume that M is the maximum among them. Since two numbers next to it in the clockwise direction are not greater than M, both nonnegative, and with the distance M between them, one of them must be equal to M and the other to 0. Therefore in some place the following triple of consecutive numbers appear: $(M, M, 0)$ or $(M, 0, M)$. From this, going along the circle counterclockwise, we can determined all other numbers, based on the rules given in the problem. In this way we obtained repeated triple $(M, M, 0)$ (so the number of points must be divisible by 3 in order that such configuration of numbers exists). In the case of 12 numbers we have 4 such triples, so it has 8 Ms and 4 zeros. From the condition that the sum is equal to 1 we get $M = \frac{1}{8}$ and so the sum of cubes squares is $8M^3 = \frac{1}{8^2} = \frac{1}{64}$.

Answer $\dfrac{1}{64}$.

20. On an exam, Question 4, Question 5, and Question 11 prove to be exceptionally difficult. 38% answered Question 4 correctly and 31% answered Question 5 correctly. 46% of students answered all three questions incorrectly and only 6% of students answered all three questions correctly. There were 18% of students who answered both Question 4 and Question 5 correctly. If 9% of students answered only Question 4 correctly (missing the other two questions) and 13% answered only Question 5 correctly, how many students answered only Question 11 correctly?

Solution

This is the flow through a 3 circled Venn Diagram.

All 3 questions correct is 6%.

Question 4 and Question 5 Correct and Question 11 incorrect is $18\% - 6\% = 12\%$.

Question 4 and Question 11 Correct and Question 5 incorrect is $38\% - 6\% - 12\% - 9\% = 11\%$.

Question 5 and Question 11 Correct and Question 4 incorrect is $31\% - 6\% - 12\% - 13\% = 0\%$.

Only Question 5 Correct $100\% - 46\% - 9\% - 12\% - 11\% - 6\% - 13\% - 0\% = 3\%$.

Only Question 11 Correct is 3%.

Answer 3%.