Math Contest CD Exam
November 12, 2022

Directions: If units are involved, include them in your answer.

1. Find $a + b + c$ if a, b, and c are positive integers satisfying

$$abc + 2ab + 2bc + 2ca + 4a + 4b + 4c = 447$$

2. Find the difference between the maximum and the minimum of y satisfying

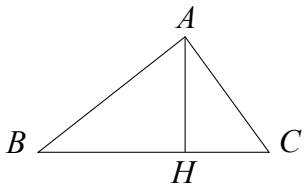
$$\log_2 x + \frac{12}{\log_2 x} - \log_x y = 6$$

if $2 \leq x \leq 16$.

3. A parallelogram has sides of length 2 and 3. One of its diagonals has length 4. Find the length of the other diagonal.

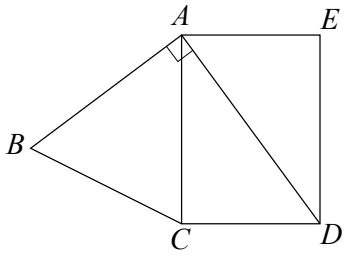
4. If $8^x - 8^{-x} = 4$ for a real number x, what is the value of $2^x - 2^{-x}$?

5. Find the area of $\triangle ABC$ if the perimeter is 30, $AH = 6$, $\overline{AH} \perp \overline{BC}$, and $\overline{AC} \perp \overline{AB}$.



6. Suppose T is a triangle in the plane with sides of length 2, 3 and 4. Let F be the figure that consists of all points of T as well as all points at distance at most 1 from the triangle. Find the perimeter of the figure F.

7. Consider a rectangle $ACDE$ with $AE = 1$, $AC = \sqrt{3}$. Let B be the point such that $AB = AC$ and $\overline{\mathrm{AB}} \perp \overline{\mathrm{AD}}$. Find the distance between C and $\overline{\mathrm{BD}}$.



8. Find the maximum of $2^x \cdot 4^y$ provided $\begin{cases} x + 3y \leq 5 \\ 2x + y \leq 5 \\ 0 \leq x,\ 0 \leq y \end{cases}$

9. The hypotenuse of the right triangle has length c and legs have length a and b. On each side of the triangle a square is drawn outside of the triangle. Express the area of the hexagon in terms of a, b, and c, whose vertices are the vertices of these squares which are not vertices of the original triangle?

10. For $0 < x < 5$ and $0 < t$, find the minimum value of the following.

$$(x - t)^2 + \left(\sqrt{25 - x^2} - \frac{72}{t}\right)^2$$

11. Let a, b, and c be three numbers (not necessarily different) chosen randomly and independently from the set $\{1, 2, 3, 4, 5\}$. Find the probability that the number $ab + c$ is even.

12. Let f be a monic polynomial of degree 4 with integer coefficients, and let $g(x) = (x - n)f(x)$ for an integer n. Find n if

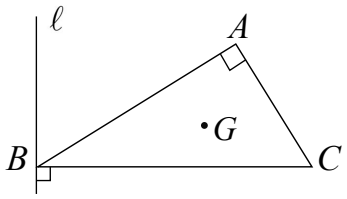
I $g(4) = 13$, $g(9) = 8$
II $f(-x) = f(x)$

13. How many 9's are there in the decimal expansion of 99999899999^2 ?

14. Find Find $f(x)$ if $f(2022x + f(0)) = 2022x^2$ for all real numbers x and $f(0) \neq 0$.

15. Suppose a rectangular prism is built out of $9 \times 13 \times 5$ unit cubes. Find the number of unit cubes that the main diagonal passes through.

16. Consider the triangle $\triangle ABC$ with $AC = 6$, $AB = 8$, and $\overline{\mathrm{AC}} \perp \overline{\mathrm{AB}}$. Let ℓ be the line passing B that is perpendicular to $\overline{\mathrm{BC}}$. Find the distance between ℓ and the centroid G of $\triangle ABC$. (The centroid of $\triangle ABC$ is the point in which the three medians of the triangle intersect)



17. Let X be the set of 8 vertices of a unit cube. Find the number of one-to-one functions $f : X \to X$ such that the distance between $f(v)$ and v is 1 for all vertices of X.

18. Let A, B, C, D, and E be the points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 2)$ respectively. Find the volume of the polyhedron with edges $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, $\overline{BE}$, $\overline{CD}$, $\overline{CE}$, and $\overline{DE}$.

19. Let α, β, and γ be the three roots of $x^3 - x - 2 = 0$. Find $\Big((\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)\Big)^2$.

20. Suppose a bike has wheels with radius 1 ft and the axle distance 3 ft. Consider two rim points on the front and rear wheels of a bike respectively with angle difference $90^\circ \pmod{360^\circ}$. Find the largest distance between these two points while a bike moves along straight line.