

Freshman-Sophomore Contest 2007  
first-year students

There are six problems. The first three problems are common to both the freshman and sophomore problem sets, and come from material studied in Math 151 and 152 or equivalent. The second page varies, so that freshmen work on three further problems from that same content, while sophomores work on three problems taken from Math 253 and Math 308 or equivalent courses.

Solutions are more than answers. Give supporting reasons for what you have done when the steps are not just mechanical, and when they are, give the intermediate steps.

1. Find

$$\int_{x=1}^2 \ln\left(x + \frac{1}{x}\right) dx.$$

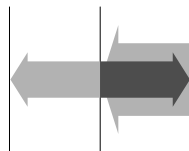
It's  $-1 - 3\ln(2) - \pi/2 + 2\ln(5) + 2\tan^{-1}(2)$ . Integration by parts does the trick.

2. Find

$$\lim_{x \rightarrow \infty} \left( \frac{\int_x^{2x} (1+t^3)^{-1} dt}{\int_{3x}^{4x} (1+t^3)^{-1} dt} \right).$$

It's  $108/7$ . Both the numerator and denominator tend to zero as  $x \rightarrow \infty$ , because the integrand is less than  $1/t^3$  so the integrals are less than what one would get integrating  $t^{-3}$  over those intervals. L'Hospital's rule applies, and in the limit,  $1/(1+t^3)$  is asymptotic to  $1/t^3$  so one can replace the former with the latter. This simplifies the calculations and results in  $(-1 + 3/4)/(-1/16 + 1/9) = 108/7$ .

3. Two flat half-silvered mirrors are lined up parallel to each other, one inch apart. Each mirror reflects half the light that falls on it, and passes half straight on through. A laser beam strikes the first mirror head-on. Half the beam thus passes through the first mirror, and half of that passes immediately through the second. Some of the light, however, bounces around between the two half-silvered mirrors before exiting either back through the first mirror or on through the second. All in all, what fraction of the beam eventually exits out the far end of the second mirror? (The figure shows the beginnings of the story, taking into account just what happens the first time the beam hits a mirror.)



One third of the light gets through. On the first pass, one quarter gets through, and one quarter bounces back toward the first mirror. Of that, half goes back into the gap, and of that half, half goes through mirror 2 the second time, so half of half of one quarter makes 1 part in 16. Continuing in this fashion, the fraction of the beam that goes out the far end of mirror two on the first, second, third etc. pass is  $1/4, 1/16, 1/64,$  and so on, for a total of  $\sum_{n=1}^{\infty} 4^{-n} = \frac{1}{3}$ .

4. A particle slides back and forth along a frictionless parabolic track with equation  $y = x^2$ , starting at  $(1, 1)$  with velocity zero. By conservation of energy, its speed, when it has position  $(x, y)$ , is proportional to  $\sqrt{1 - y} = \sqrt{1 - x^2}$ . There is a point along the parabola such that, as it passes that point, the downward component of its velocity is maximum. Find that point.

Whatever the velocity is, it resolves into part downward, part sideways. The component of any particular mph of speed that is downward is given by  $2x/\sqrt{1 + 4x^2}$  while the component that is sideways is  $1/\sqrt{1 + 4x^2}$ . So we have to maximize  $2x\sqrt{1 - x^2}/\sqrt{1 + 4x^2}$ . Maximizing a square root corresponds to maximizing the inside of the square root, so the maximum will occur at whatever value of  $x$  makes  $4x^2(1 - x^2)/(1 + 4x^2)$  maximal. But  $y = x^2$ , so we might as well look for that value of  $y$  which maximizes  $4y(1 - y)/(1 + 4y)$  over the interval  $0 < y < 1$ . Taking the derivative gives a fraction in which the numerator is  $1 - 2y - 4y^2$  which will have to be zero, and this gives  $y = (\sqrt{5} - 1)/4$ . That, finally, gives  $x = \sqrt{(\sqrt{5} - 1)/4}$ , and the required point has been found.

5. Find the radius of convergence of the following power series':

(a)  $\sum_{n=1}^{\infty} e^{-n} x^n,$

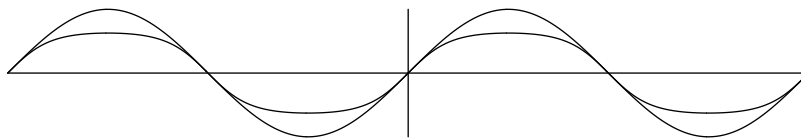
(b)  $\sum_{n=1}^{\infty} e^{-\sqrt{n}} x^n,$

- (c)  $\sum_{n=1}^{\infty} e^{-n\sqrt{n}} x^n.$  The radius for the first one is  $e$  by the ratio test. For the second, the ratio is, in the limit,  $x$ , so again by the ratio test, the radius is 1. For the last, the ratio is

$$xe^{-(n+1)\sqrt{n+1}+n\sqrt{n}} < xe^{-(n+1)\sqrt{n}-n\sqrt{n}} = xe^{-\sqrt{n}}.$$

The limit, for fixed  $x$  as  $n \rightarrow \infty$ , is thus zero and so the radius of convergence is  $\infty$ .

6. The graphs of  $\sin(x)$  and  $\sin(\sin(\sin(\sin(\sin(x))))))$  are shown:



How to account for this? Let  $s_n(x)$  denote the result of starting with  $x$  and applying the sine function  $n$  times. Thus  $s_1(x) = \sin(x)$ ,  $s_2(x) = \sin(\sin(x))$ ,  $s_3(x) = \sin(\sin(\sin(x)))$ , and so on. Thus  $s_{n+1}(x) = \sin(s_n(x))$ . Each of the parts below is to be proved for all  $n$ , and either for all  $x$ , or for all  $x$  in a given interval. You will need to use induction for some of the proofs. For each part, you are allowed to take as given all the previous parts, whether or not you found your own proof for that previous part.

- (a) Prove  $s_n(-x) = -s_n(x)$ . We have  $\sin(-x) = -\sin(x)$ . Now if the statement holds for  $n$ , then for  $n + 1$ , we have

$$s_{n+1}(-x) = \sin(s_n(-x)) = \sin(-s_n(x)) = -\sin(s_n(x)) = -s_{n+1}(x).$$

By induction, it holds for all  $n$ .

- (b) Prove  $s_n(x + \pi) = -s_n(x)$ . We have

$$s_n(x + \pi) = s_{n-1}(\sin(x + \pi)) = s_{n-1}(-\sin(x)) = -s_{n-1}(\sin(x)) = -s_n(x)$$

where the next to last step was based on part (a) above.

- (c) Prove  $s_n(x + 2\pi) = s_n(x)$ . Apply the previous part twice.  
 (d) Prove  $s_n(\pi - x) = s_n(x)$ . From parts (b) and then (a), we have  $s_n(-x + \pi) = -s_n(-x) = s_n(x)$ .  
 (e) Prove  $s_n(x)$  is positive for  $0 < x < \pi$ . This is true for  $n = 1$ . Now assume it holds for  $n$ , and consider  $s_{n+1}(x) = s_n(\sin(x))$ . For  $0 < x < \pi$ ,  $0 < \sin(x) < 1 < \pi$ , and so  $s_n(\sin(x))$  is an instance of  $s_n$  applied to a number between 0 and  $\pi$ , so we get a positive result. Thus by induction, the statement that  $s_n(x)$  is positive for all  $x$  in  $(0, \pi)$  is proved.  
 (f) Prove  $s_n(x)$  is increasing on the interval  $0 < x < \pi/2$ . Again we proceed by induction. This is true for  $n = 1$ . Consider

$$s'_{n+1}(x) = \frac{d}{dx} (\sin(s_n(x))) = \cos(s_n(x)) \frac{d}{dx} s_n(x).$$

Both factors here are positive. The first is positive because  $s_n(x)$  is positive, but it's also the sine of something so it's no more than 1, so  $0 < s_n(x) \leq 1$  and so  $0 < s_n(x) < \pi/2$  so  $\cos(s_n(x)) > 0$ . The second is positive by the inductive assumption.

- (g) Prove  $s_n(x)$  is decreasing on the interval  $\pi/2 < x < \pi$ . This follows from the previous part together with part (d).  
 (h) Prove  $s_n(\pi/2) \geq 1/\sqrt{n}$ . For  $n = 1$ , we have equality. Now we just need to prove that  $\sin(\sqrt{1/n}) > \sqrt{1/(n+1)}$ . For  $n = 1$ , this is true because  $\sin(\pi/4) = 1/\sqrt{2}$  and  $\sin(1) > \sin(\pi/4)$  because  $1 > \pi/4$ . For larger values of  $n$ , we have

$$\sin(n^{-1/2}) = n^{-1/2} \left( 1 - \frac{1}{3!n} + \frac{1}{5!n^2} + \dots \right)$$

while

$$(n+1)^{-1/2} = n^{-1/2} \left( 1 - \frac{(1/2)}{n} + \frac{(1/2)(3/2)}{2!n^2} - \frac{(1/2)(3/2)(5/2)}{3!n^3} + \dots \right)$$

The second series is smaller than the first because the difference is an alternating series starting with  $1/(3n)$  and the terms decrease in absolute value.