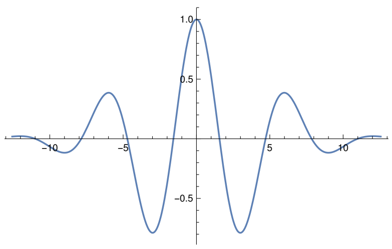

Figure 1: Graph of damped cosine wave

Solutions for the Texas A&M Freshman-Sophomore Contest 2017

Second year student version

There are two pages, six problems. The first three problems are common to both versions.

1. Let $f(x) = \cos(x)e^{-x^2/(4\pi^2)}$.

   (a) Sketch the graph of $f(x)$ over the interval $[-4\pi, 4\pi]$.

   (b) Find the derivative of $f(x)$ at $x = \pi$ and simplify fully. The product rule and the chain rule come into play because f is the product of $\cos x$ and $e^{-x^2/4\pi^2}$. The derivative works out to $(-x\cos x/(2\pi^2) - \sin x)e^{-x^2/4\pi^2}$. Setting $x = \pi$ zeroes out the sine term and the answer is $\frac{1}{2\pi}e^{-1/4}$.

2. The identity $\cos(2t) = 2\cos^2(t) - 1$ has some curious consequences.

   (a) Express $\cos(4t)$ in terms of $\cos t$. It's $8\cos^4(t) - 8\cos^2 t + 1$.

   (b) Sketch the graph of $y = x^4 - x^2 + \frac{1}{8}$ on the interval $-1 \le x \le 1$, and find the minimum value of y and where it occurs. The minimum value is $-1/8$ because of the first two parts, which imply that this polynomial is $(1/8)\cos(4\cos^{-1} x)$. The minimum value occurs at $x = \pm 1/\sqrt{2}$ because the derivative is $2x(2x^2 - 1)$ which is zero at those places and at zero. But at 0, the original polynomial evaluates to positive. Because of the tie-in with the cosine function, the graph runs back and forth between $-1/8$ and $1/8$; the maximum occurs at 0 and at ± 1, while the minimum occurs at $\pm 1/\sqrt{2}$. (Polynomials that agree on $[-1, 1]$ with $\cos(n\cos^{-1} x)$ are called *Chebyshev* polynomials and have all sorts of interesting properties, not just the one that defines them.)

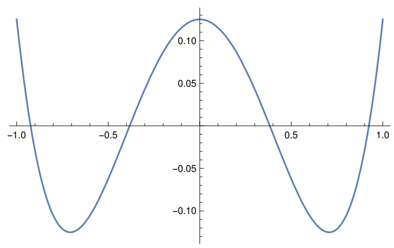

Figure 2: Graph of Cheybshev-type polynomial

3. Take as given the power series expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Find in closed form

$$A = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \frac{(2k-1)!}{(k-1)!2^{2k-1}}.$$

The $(2k-1)!$ in the numerator cancels all of the $(2k)!$ in the denominator except for its final factor $2k$. Putting the 2 here with the 2^{2k-1} gives that

$$A = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{-2k}}{k!} = e^{-1/4},$$

this last by the series expansion for e^z specialized to the case $z = -1/4$.

4. A triangular field is overgrown with tumbleweeds. They're all over the field, as thick in one part of the field as in any other.

   It's a right-triangle shape, with the right-angle corner at Zed's Crossroads. The field extends North 1 mile along 0th avenue, and one mile East along 0th street, and the third side, the hypotenuse, cuts across at a 45 degree angle to both roads to complete the triangle. The field thus has an area of half a square mile.

   It's fenced on all sides. One night there's a storm, with a strong West wind. All the tumbleweeds break loose and roll across the field, heading East, until they hit the fence. What is the average distance traveled by a tumbleweed that night?

   The average value of a quantity in a region is the integral of that quantity divided by the area (or length, or volume, as the case may be) of the region. Here, the region is a triangle with area 1/2 (in square miles) and the quantity is given by the function of (x, y) which gives the distance to the fence going East. Along the fence, $y = 1 - x$, so the distance traveled by a tumbleweed that starts at (x, y) inside the field is $(1 - y - x)$ (as the tumbleweed drifts, x increases from whatever it was originally to a final value of $(1 - y)$). So the answer will be given by

$$A = 2\int_{x=0}^{1}\int_{y=0}^{1-x}(1-x-y)\,dy\,dx$$
$$= 2\int_0^1\left((1-x)y - \frac{1}{2}y^2\right)\Big|_0^{1-x}\,dx = 2\int_0^1((1-x)^2 - \frac{1}{2}(1-x)^2)\,dx = \frac{1}{3}.$$

The average tumbleweed travels 1/3 of a mile before hitting the fence.

5. Find

$$\int_B \frac{x^2}{x^2+y^2+z^2}^{5/4}\,dV$$

where B is the solid ball $x^2 + y^2 + z^2 \leq 1$. By symmetry, the integral would be the same if the numerator were y^2 or z^2 instead of x^2. Thus, the original integral is

$$\frac{1}{3}\int_B (x^2+y^2+z^2)^{-1/4}\,dV.$$

Passing to spherical coordinates, this becomes

$$\frac{1}{3}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{rho=0}^{1}(\rho)^{-1/2}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

The inner integral evaluates to $(2/15)\sin\phi$. Next, $\int_0^\pi(2/15)\sin\phi\,d\phi = 4/15$. Finally, $\int_0^{2\pi}(4/5)\,d\theta = 8\pi/15$. The answer is $8\pi/15$.

Without the help of this symmetry the technical calculations become more difficult. The spherical-coordinates version of the integral becomes

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{rho=0}^{1}(\rho\sin\phi\cos\theta)^2\rho^{-5/2}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Once again one must integrate $\rho^{3/2}$, but then one must work through integrating $\sin^3\phi\cos^2\theta$. Here, the trigonometric identities $\cos^2\theta = (1 + \cos\theta)/2$ and $\sin^3\phi = \sin(\phi)\sin^2\phi = \sin(\phi)(1-\cos^2\phi)$ come in handy. This last one allows for a substitution $u = \cos(\phi)$ that turns the trigonometric integral into an integral along the lines of $\int_0^1(1-u^2)\,du$. The upshot is again $8\pi/15$, as it must be because correct solutions, be they elegant or straightforward, must in the end arrive at the same answer.

6. Consider an 8×8 square $S = [0, 8] \times [0, 8]$. A knight can occupy any point of the square; the coordinates don't have to be whole numbers. The knight can move to any point inside the square such that one coordinate of that new point differs by 2 from the old coordinate, and the other differs by 1.

Let $F(x, y)$ be the number of points to which the knight can move from (x, y). Thus $F(0.3, .2.1) = 4$ because the points available to a knight there are $(1.3, 0.1)$, $(1.3, 4.1)$, $(2.3, 1.1)$, and $(2.3, 3.1)$.

Find $\int_S F(x, y)\, dy\, dx$. $F(x, y)$ can be seen as the sum of eight simpler functions, each taking the value 1 if a particular knight move is legal and 0 otherwise. For each of these eight functions, the region where it takes the value 1 is a rectangle 6 by 7, so the integral of each of them is just the area of that rectangle. Eight copies of 42 give an integral of 336.