TAMU Freshman-Sophomore Contest, 2016

First-year students' version

While the name of the contest is traditional, the actual eligibility rules are that first year students take the freshman contest, and second year students take the sophomore contest. That way, students who have accumulated enough credit hours in their first or second year to have standing as sophomores, or juniors, are not promoted out of eligibility.

The first page contains problems built around Calculus I and II for both freshmen and sophomores. The second page is again pitched to content particular to Calculus I and II, in the case of first-year students.

In all cases, solutions should be written out and should include reasoning behind the steps when reasons beyond routine calculation are involved. No tables, calculators, or computers, and no devices for communication with the outside world, are allowed. You're on your own.

1. Find $\int_0^{\pi/2} \cos x \cos 2x \, dx$. The identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$, applied to the cases $a = 2x, b = x$ and $a = 2x$, $b = -x$ yields $\cos 2x \cos x = \frac{1}{2}(\cos 3x + \cos x)$. Integrating this from 0 to $\pi/2$ gives an answer of $\frac{1}{2}(\sin 3x/3 + \sin x|_0^{\pi/2} = \frac{1}{3}$. The answer is $1/3$.

   For another solution, the same identity for $\cos(a+b)$ is applied instead to the case $a = b = x$, so that $\cos x \cos 2x = (\cos x)(\cos^2 x - \sin^2 x)$. But now

   $$\int_{x=0}^{\pi/2} (\cos x)(\cos^2 x - \sin^2 x)\, dx = \int_{x=0}^{\pi/2} (1 - 2\sin^2 x) \cos x \, dx$$
   $$= \int_{u=0}^{1} (1-u^2)\, du = 1 - \frac{2}{3} = \frac{1}{3}$$

   by way of the substitution $u = \sin x$, $du = \cos x \, dx$. Which is the better solution? From one perspective, the second solution is best. It's shorter and easier to understand. From another perspective, the first solution is best because it can be adapted to a wider variety of similar cases, such as $\cos(2x)\cos(3x)$.

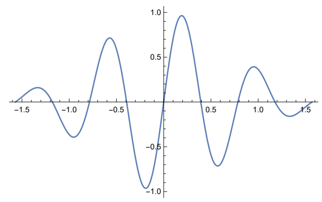
2. Let $f(x) = \frac{x}{1+x^2}$. Let $g(x)$ be the 19th derivative of $f(x)$. Find $\frac{g(0)}{20!}$. Probably the best way to work this is to first get the series expansion of $f(x)$ about 0. The series $1/(1-z) = 1 + z + z^2 + \cdots$, with $z = -x^2$, gives $\frac{x}{1+x^2} = x - x^3 + x^5 - x^7 \cdots + x^{17} - x^{19} + \cdots$. Taking the 19th derivative term by term, as we may do inside the radius of convergence, which is 1 here, gives

   $$f^{(19)}(x) = -19! + \frac{21!}{2!}x^2 - \frac{23!}{4!}x^4 - \cdots$$

   and setting $x = 0$ and dividing by $20!$, the answer is $-1/20$.

3. Let $u(x) = \sin(8x)e^{-x^2}$.

   (a) Graph $u(x)$ on the interval $-\pi/2 \le x \le \pi/2$.



   (b) Given that $\int_0^\infty t^k e^{-t}\, dt = k!$ for all nonnegative integers k, prove that

$$\int_0^\infty u(x)\, dx = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{k!8^{2k+1}}{(2k+1)!}.$$

Here we need to break up $\sin(8x)$ into its power series, but not break up e^{-x^2}. A truly careful proof must address matters of convergence, and that can be done here by cutting off the series at some point N and then proving that the rest of the series is at any rate bounded between what one would get with all the terms positive from then on, and with all of them negative, and that both bounds tend to zero. For all x, $\sin(8x) = \sum_{k=0}^{\infty}$. For any positive integer N, this can be split as $\sin(8x) = (\sum_{k=0}^{N} + \sum_{k=N+1}^{\infty}(-1)^k(8x)^{2k+1}/(2k+1)!$. Let the first sum be $S_N(x)$ and the second sum be $R_N(x)$, as in sum and remainder. We have $|R_n(x)| \le \sum_{k=N+1}^{\infty}(8x)^{2k+1}/(2k+1)!$. Now let's get down to the mechanics.

$$\int_0^\infty S_N(x)\, dx = \int_0^\infty \sum_{k=0}^{N}(-1)^k 8^{2k+1}x^{2k+1}e^{-x^2}/(2k+1)!\, dx$$

$$= \sum_{k=0}^{N}\left((-1)^k 8^{2k+1}/(2k+1)!\right)\int_{x=0}^{\infty} x^{2k+1}e^{-x^2}\, dx$$

$$= \frac{1}{2}\sum_{k=0}^{N}\left((-1)^k 8^{2k+1}/(2k+1)!\right)\int_{u=0}^{\infty} u^k e^{-u}\, du$$

$$= \frac{1}{2}\sum_{k=0}^{N}(-1)^k 8^{2k+1}\frac{k!}{(2k+1)!}.$$

Also, $\int_0^\infty R_N(x)\,dx < \sum_{k=N+1}^\infty 8^{2k+1}\frac{k!}{(2k+1)!}$, because with everything positive, we can switch summation and integration with confidence. If any version converges then the others do as well, and all of them, with absolute convergence. For $N > 10$, say, each term is less than half the one before it in this sum, so the total is less than twice the first term. That is, $\int_0^\infty R_N(x)\,dx < 2\cdot 8^{2(N+1)+1}\frac{(N+1)!}{2N+3)!}$, an expression which tends to 0 rapidly as N tends to infinity because the factorial of $2N+3$ dominates.

So for all N, $\int_0^\infty u(x)\,dx = \int_0^\infty S_N(x) + R_N(x)\,dx$. The first integral evaluates to an expression whose limit is the claimed answer, and the second integral evaluates to an expression whose limit is 0. We are done.

4. Find
$$\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{3/2}}.$$
With $t = \tan\theta$, the integral becomes $\int_{\theta=-\pi/2}^{\pi/2} \frac{\sec^2\theta}{(1+\tan^2\theta)^{-3/2}}\,d\theta$. But this simplifies to $\int_{\theta=-\pi/2}^{\pi/2}\cos(\theta)\,d\theta = 2$. So the answer is 2.

5. Let $\langle a_n\rangle$ be the sequence given by $a_0 = 0$, $a_1 = 1$, and $a_n = 2a_{n-1} + a_{n-2}$.

   (a) Find a_6. It's 70. On the way, $a_2 = 2$, $a_3 = 5$, $a_4 = 12$, and $a_5 = 29$.

   (b) Taking as given, for the time being, that the limit exists, find
   $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}.$$
   Say L is the limit. Then eventually, a_{n+1}/a_n and a_n/a_{n-1} are both arbitrarily close to L. So $a_{n+1}/a_n - a_n/a_{n-1}$ is close to 0. But $a_{n+1}/a_n = 2 + a_{n-1}/a_n$, so L is close to $2 + 1/L$. Arbitrarily close. But that means $L = 2+1/L$. Multiply the equation by L and use the quadratic formula to conclude that $L = (2\pm\sqrt{8})/2 = 1\pm\sqrt{2}$. But L must be positive because all the a_n are positive. Thus $L = 1+\sqrt{2}$.

   (c) Prove that the limit exists. Begin with some simple observations: first, that $a_{n+1} > a_n$ and so $\langle a_n\rangle \to \infty$ as $n \to \infty$. Next, a look at the first few numbers and some tinkering discloses a pattern: $a_{n+1}a_{n-1} - a_n^2 = (-1)^n$. This requires proof to be used, so here's a proof by induction.

   The statement holds for $1 \le n \le 5$ by direct arithmetic. (For instance, when $n = 5$, $70 * 12 - 29^2 = 840 - 841 = -1 = (-1)^5$.) Now suppose it holds for $1 \le n \le N$. We have to prove it also holds for $n = N+1$. That is, we must show that $a_{N+2}a_N - a_{N+1}^2 = (-1)^{N+1}$. We have $a_{N+2} = 2a_{N+1} + a_N$, and $a_{N+1} = 2a_N + a_{N-1}$ and $a_{N+1}a_{N-1} - a_N^2 = (-1)^N$. Thus
   $$\begin{aligned} a_{N+2}a_N - a_{N+1}^2 &= (2a_{N+1} + a_N)a_N - aN + 1(2a_N + a_{N-1}) \\ &= a_N^2 - a_{N+1}a_{N-1} = -(-1)^N. \end{aligned}$$

The induction is now complete. Now consider the sequence of differences $a_{n+1}/a_n - a_n/a_{n-2}$. This simplifies to $(-1)^n/(a_n a_{n-1})$. But that's an alternating sequence whose terms decrease to zero. The sequence a_{n+1}/a_n goes up some, then down less, then up still less, and so on, forever. An increasing, bounded sequence such as $\langle a_{2n}/a_{2n-1}$ must have a limit, and a decreasing, bounded sequence such as $\langle a_{2n+1}/a_{2n}$ also has a limit, and since the two sequences are separated by an amount that tends to zero, those two limits are the same, both equal to L.

6. Let $A_0 = 1, B_0 = 2$. For $n \geq 1$, let

$$A_n = \sqrt{A_{n-1}B_{n-1}}, \qquad B_n = \tfrac{1}{2}\left(A_{n-1} + B_{n-1}\right).$$

   (a) Find B_2. It's $\sqrt{2}/3/4$.

   (b) Prove that for $n \geq 1$, $B_n - A_n < (B_{n-1} - A_{n-1})^2$. Proof. It will be sufficient to prove that if $A, B \geq 1$ then $((A+B)/2 - \sqrt{AB}) < (B-A)^2$. But

$$((A+B)/2 - \sqrt{AB})((A+B)/2 + \sqrt{AB}) = \frac{1}{4}(B^2 + 2AB + A^2) - AB$$
$$= \frac{1}{4}(B-A)^2.$$

   Since $(A+B)/2 + \sqrt{AB} \geq 2$,

$$((A+B)/2 - \sqrt{AB}) \leq (1/8)(B-A)^2.$$

   (c) Make a reasoned guess as to the least n such that $B_n - A_n < 10^{-100}$. Explain your thinking.

   Some arithmetic is needed to get started. We have $A_1 = \sqrt{2} = 1.414$, roughly, and $B_1 = 1.5$, so $B_1 - A_1 < 1/10$. Now $B_2 - A_2$ will be less than 1/800, but not far less, because that factor of 1/8 cannot be improved to 1/16 if you go back into the proof of the other problem item.

   So we expect something like $B_3 - A_3 = 10^{-3}$, then $B_4 - A_4 = 10^{-7}$, then $B_5 - A_5 = 10^{-15}$, then $B_6 - A_6 = 10^{-31}$, then $B_7 - A_7 = 10^{-63}$, and then $B_8 - A_8 = 10^{-127}$. So 7 or 8 would be reasonable guesses. (A more accurate calculation shows that in reality it's 7, rather than 8. A guess of 9 would not have been altogether unreasonable. That's what you'd think going merely on the answer to the previous part. The moral of the story is that it often pays to use not just a result, but to look inside the result at the proof and use that.