2019 Power Team
Texas A&M High School Students Contest
November 2019

The Discrete Universe

Try to solve as much as you can. If you did not succeed to explore the most general case, but were able to treat some particular case, please describe all your findings. In all problems prove your answers.

The goal of this set of problems is to explore the motion of particles in a world that consists of a discrete set of points, for example, points with integer coordinates on a line or a plane. Time in this word takes integer values. For example, assume that our universe consists of all integers on a line. If $x(t)$ is the position of the point at time t in this world, then the discrete velocity $v(t)$ at time $t \geq 1$ is defined by

$$v(t) = x(t) - x(t-1) \tag{1}$$

and the discrete acceleration $a(t)$ at time $t \geq 0$ is defined by

$$a(t) = v(t+1) - v(t). \tag{2}$$

Let $F(x)$ be an integer-valued function of a position which describes the discrete force acting on a particle of mass 1 at a position x. In this case we say the force field F acts on our discrete line and the discrete Newton second law says that if the particle of a unit mass is at the point $x(t)$ at time t, then the discrete acceleration $a(t)$ of this particle at time t satisfies:

$$a(t) = F\big(x(t)\big).$$

and the motion (the trajectory) of the particle is uniquely determined by prescribing the initial position $x(0)$ and the initial velocity $v(0)$.

Further, we say that the trajectory $x(t)$ is *periodic*, if there exists an integer T such that for every integer $t \geq 0$ we have

$$x(t+T) = x(t).$$

The minimal T satisfying this property is called the *period* of the periodic trajectory $x(t)$.

Part 1

1. Assume that the discrete force field is given by

$$F(x) = -\mathrm{sgn}(x), \text{where } \mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0, \\ -1, & x < 0 \end{cases} \tag{3}$$

   Prove that with this force field $F(x)$ any trajectory of the unit mass particle is periodic. Then find its period and amplitude as functions of the initial position and the initial velocity. Here by the *amplitude* A of the periodic trajectory $x(t)$ we mean the half distance between the maximum and minimum value of $x(t)$ for all integer $t \geq 0$.

2. The *conservation law* for a given force field $F(x)$ is any *nonconstant* function $I(x, v)$ such that

$$I\big(x(t), v(t)\big) = I\big(x(0), v(0)\big), \qquad \text{for every } t \geq 0,$$

   where $x(t)$ is any trajectory of the unit mass particle in the force field F and $v(t)$ is its velocity.

   Find at least one conservation law for the trajectories of unit mass particle in the force field of problem 1.

3. Study problems 1 and 2 under the assumption that the discrete universe is not $\mathbb{Z}$ but $\mathbb{Z} + \frac{1}{2}$, i.e. the set of all points with fractional part 1/2 (note that in this case still $v(t) \in \mathbb{Z}$). Here it is not necessary to give as detailed solution as in the problems 1 and 2. Just indicate the similarities and differences in the results and conclusions in this setting compared to the previous one.

4. Now consider the motion of two unit mass particles in the universe consisting of $\mathbb{Z}$ such that if $x_1(t)$ and $x_2(t)$ are positions at time t of the first and the second particle, respectively, then the discrete acceleration $a_1(t)$ at time t of the first particle is equal to sign $(x_2(t) - x_1(t))$ and the discrete acceleration $a_2(t)$ at time t of the second particle is equal to $\mathrm{sign}\big(x_1(t) - x_2(t)\big)$. Study problems 1 and 2 for this setting, namely

   (a) Find all initial positions and velocities of particles for which the corresponding trajectories $\big(x_1(t), x_2(t)\big)$ are periodic, i.e. there exist $T > 0$ such that

$$x_1(t+T) = x_1(t), \quad x_2(t+T)\big) = x_2(t), \quad \forall t \geq 0.$$

   For every periodic solutions find its period.

(b) Find two independent conservation laws for the given force field. Independent means that one conservation law is not a composition of a single variable function with the other conservation law.

(c) Describe all (not only periodic) trajectories of the system.

Part 2

In this part our universe consists of the vertices of a regular n-gon $P_0P_1 \dots P_n$. Let O be the center of the n-gon. The particle moves in this universe by jumping from one vertex to another vertex of the n-gon. Let $x(t)$ be the position of the particle at time t. Assume that $x(0) = P_0$ and for every time $t > 0$ the angle between the ray $Ox(t-1)$ and $Ox(t)$, counted in the counterclockwise direction, is equal to $\frac{2\pi}{n}t$. In other words, the particle moves in our universe with constant angular acceleration 1, starting at P_0 with 0 angular velocity. Further, let $A(n)$ denote the number of different vertices of the n-gon that are visited by a particle during its motion. The goal of the following exercises is to express $A(n)$ in terms of the prime factorization of n.

5. Prove that for every n the trajectory $x(t)$ is periodic and find its period $T(n)$.

6. Find $A(n)$ if

   (a) $n = 2^k$.

   (b) n is an odd prime.

   Prove your answers.

7. (a) Find a relation between $A(2n)$ and $A(n)$. Prove your answer.

   (b) Let m and n be odd and coprime. Find a relation between $A(m)$, $A(n)$, and $A(mn)$. Prove your answer.

8. Given n denote by Δ_n the set of all possible remainders obtained after the division of numbers of the form $\frac{t(t+1)}{2}$ by n (so that $A(n)$ is the number of elements of Δ_n). In the present problem you can use the following fact without proof: for $n = p^k$, where p is an odd prime, the number of elements m in Δ_n such that the number $8m+1$ is not divisible by p is equal to $\frac{p^k - p^{k-1}}{2}$. Based on this fact, find $A(n)$ if

   (a) $n = p^2$, where p is an odd prime.

(b) $n = p^k$, where $k \geq 3$ and p is odd prime.

9. Express $A(n)$ in terms of the prime factorization of n for arbitrary n.

10. Find the maximal and the minimal values among all ℓ with the following property: there exists a sequence of natural indices $\{n_k\}_{k=1}^{\infty}$ with $n_k < n_{k+1}$ such that

$$\ell = \lim_{k\to\infty} \frac{A(n_k)}{n_k}.$$