Let us denote by $f(n,t)$ the mass of the particle at coordinate n at moment t. We use the standard notation $\binom{n}{k}$ for the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}.$$

We assume that $\binom{n}{k} = 0$ if k is not an integer. We also assume that $\binom{n}{k} = 0$ for all integers $k > n$ and $k < 0$.

Problem 1. We have initially $f(n,0) = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$ The conditions of the problem are equivalent to the following recurrent rule:

$$f(n,t+1) = (f(n-1,t) + f(n+1,t))/2.$$

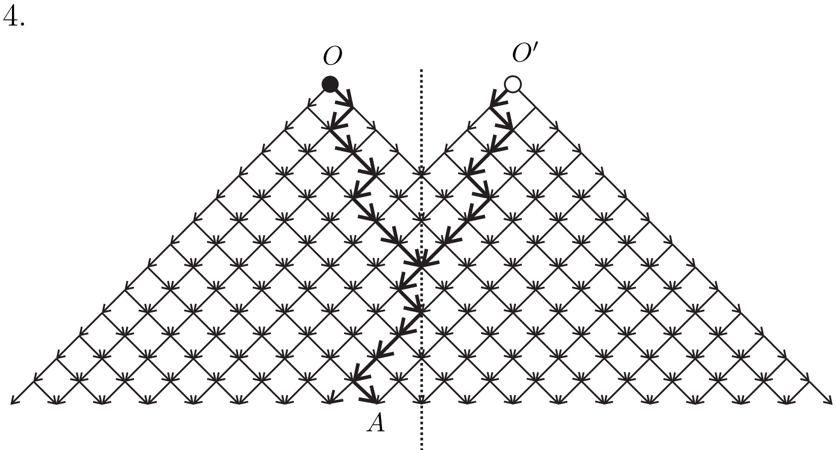
Consider the function $F_t(s) = \sum_{n=-\infty}^{\infty} f(n,t)s^n$. The recurrent rule is then equivalent to the relation $F_{t+1}(s) = \frac{s^{-1}+s}{2} F_t(s)$.

Since we have $F_0(s) = 1$, we get $F_t(s) = \frac{(s^{-1}+s)^t}{2^t}$. Therefore, $f(n,t)$ is the coefficient at s^n of $(s^{-1}+s)^t$ divided by 2^t. If we write $n = k_1 - k_2$ so that $k_1 + k_2 = t$, then we get from the binomial formula $f(n,t) = \binom{t}{k_1}$. It follows that

$$f(n,t) = \begin{cases} \frac{1}{2^t}\binom{t}{(n+t)/2} & \text{if } (n+t)/2 \text{ is a non-negative integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2. The masses of the particles in Problem 1 are equal to 2^{-t} times the numbers paths from the top black vertex O to the corresponding points A of the graph. For example, the mass $f(n,t)$ at point at coordinate $n = 2$ and moment $t = 14$ is 2^{-14} times the number of paths from O to the point A shown on the figure, since in order to reach A from O one has to make 8 steps to the right and 6 steps to the left.

The masses of the particles for Problem 2 are equal then 2^{-t} times the number of paths from O to A that do not touch the vertical line placed at coordinate k, like the one drawn on the following figure for $k = 4$.



For every path from O to A that touches the line, take the first point P when the path is on the line and reflect the part OP of the path with respect to the vertical line. We will get a path starting in the vertex O' at coordinate $2k$. Note that every path starting in O' and ending to the left side of the vertical line must touch the vertical line. We get for every point A with coordinate $\leq k$, a one-to-one correspondence between the set of paths from O' to A and the set of paths from O to A that touch the vertical line at coordinate k.

It follows that the number of paths from O to A not touching the vertical line is equal to the number of paths from the O to A minus the number of paths from O' to A. Therefore, the answer to the problem is

$$2^{-t}\binom{t}{(n+t)/2} - 2^{-t}\binom{t}{(n+t)/2-k}$$

if $(n+t)/2 \in \mathbb{Z}$ and $n \leq k$, and 0 otherwise.

Another interpretation of the solution. Put at the initial moment a particle of mass 1 in the point O with coordinate 0 and a particle of mass -1 in the point O' with coordinate $2k$. The rules of the evolution is the same as before.

Since the rules and the initial condition is symmetric with respect to the reflection with respect to the coordinate k (where the reflection also changes the sign of mass), the mass of the particle at coordinate k is always equal to zero. It follows that if we restrict the function $f(n,t)$ of the mass to the points of coordinates $n \leq k$, then the rule of the change of mass is exactly as in Problem 2. But then it is clear that the mass of a point A at moment t is 2^{-t} times the number of paths from O to A minus the number of paths from O' to A, like in the first solution.

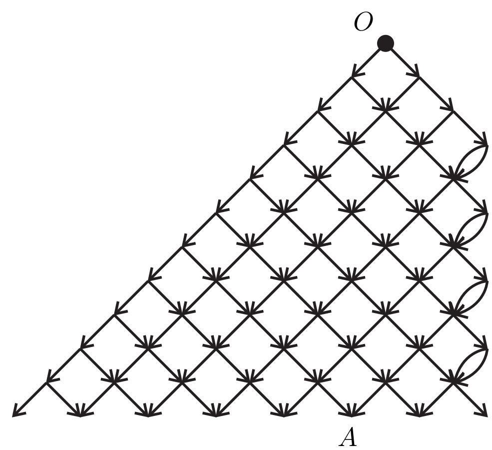
Problem 3 We are using the ideas of the second solution of the previous problem but now we put two "mirrors": one at coordinate k and another at coordinate l. Namely, we put particles of masses 1 in points of coordinates of the form $2mk - 2ml$ for all $m \in \mathbb{Z}$, and particles of masses -1 in points of coordinates $2(m+1)k - 2ml$ for all $m \in \mathbb{Z}$. Reflections with respect to k and l are $x \mapsto 2k - x$ and $x \mapsto 2l - x$, respectively. A point $2mk - 2ml$ is mapped by the first reflection to $2k - 2mk + 2ml = 2(-m+1)k - 2(-m)l$ and by the second one to $2l - 2mk + 2ml = 2(-m)k - 2(-m-1)l$. A point $2(m+1)k - 2ml$ is mapped by the first reflection to $2k - 2(m+1)k + 2ml = 2(-m)k - 2(-m)l$ and by the second one to $2l - 2(m+1)k + 2ml = 2(-m-1)k - 2(-m-1)l$. We see that both reflections move the set of particles of mass 1 to the set of particles of mass -1 and vice versa. In particular, the masses of the particle at the points k and l are always equal to zero.

Consequently, by the same arguments as in Problem 2, we get that the answer can be written as

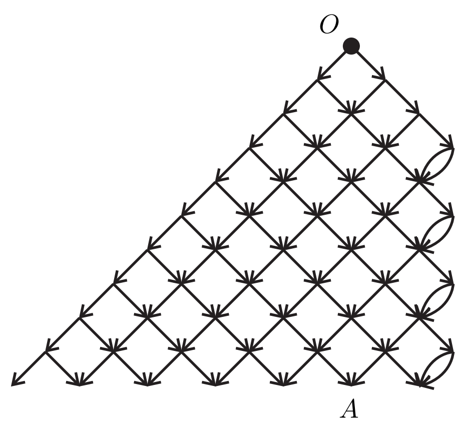
$$2^{-t}\sum_{m\in\mathbb{Z}}\binom{t}{\frac{n+t}{2}-mk+ml}-2^{-t}\sum_{m\in\mathbb{Z}}\binom{t}{\frac{n+t}{2}-(m+1)k+ml}$$

Note that both sums are actually finite for every value of (n,t).

Problem 4. We will again use the method of "trajectories" from the previous problems. Reflective screen can be interpreted as counting paths in a graph of the form

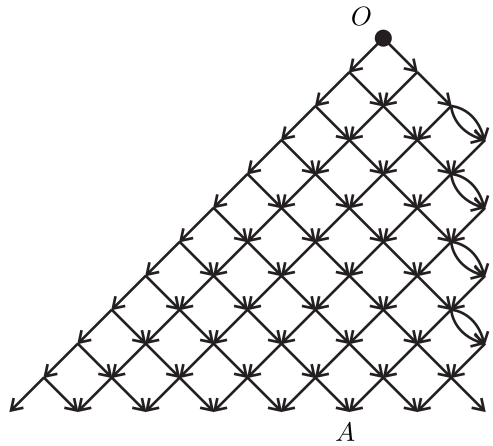


or

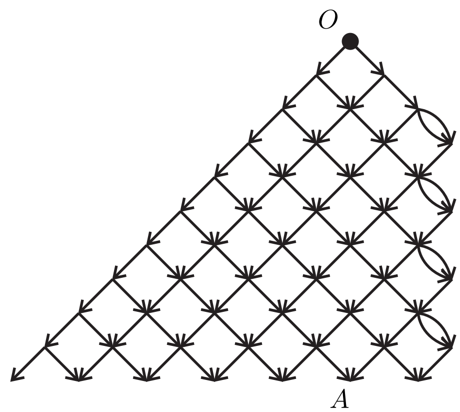


depending on parity of t. The answer to the problem will be again 2^{-t} times the number of paths from O to the corresponding point A.

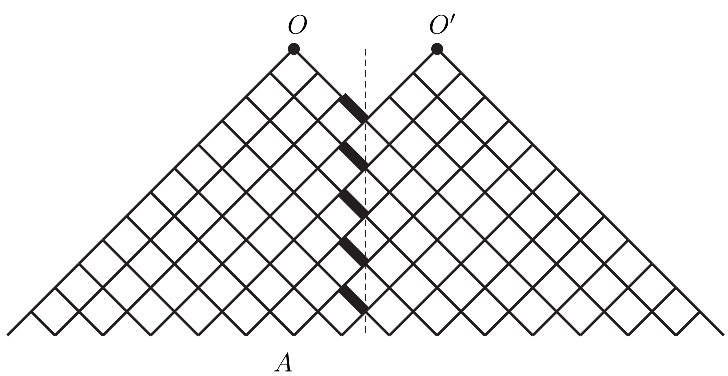
We can replace the above graphs by the graphs



and



Consider now the following graph, where O' is the point at coordinate $2k$.



The function $F(A)$ on the set of vertices A equal to the number of paths from the set $\{O, O'\}$ to A (where, as before, the edges are directed down) is symmetric with respect to the dashed line (at coordinate k). In particular, the value of F at the lower end of the highlighted edges are double of the value at their higher end. It follows that $F(A)$ for A to the left of the axis of symmetry is equal to the number of paths in the two graphs on the previous figures, except for the case when the vertex A is on the axis, when the number of paths is $F(A)/2$.

It follows that the answer to Problem 4 is

$$\begin{cases} 2^{-t}\left(\binom{t}{(n+t)/2}+\binom{t}{(n+t)/2-k}\right) & \text{if } (n+t)/2 \in \mathbb{Z} \text{ and } n<k, \\ 2^{-t}\binom{t}{(k+t)/2} & \text{if } (n+t)/2 \in \mathbb{Z} \text{ and } n=k, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 5. Let $f(n,t)$ be the answer to Problem 1, and let $g(n,t)$ be the answer to Problem 4. We have then $f(0,0)=g(0,0)=1$ and $f(n,0)=g(n,0)=0$ for all $n \neq 0$. The corresponding conditions of the problems also give us

$$f(n,t+1)=\frac{1}{2}(f(n-1,t)+f(n+1,t))$$

for all n, and

$$g(n,t+1)=\frac{1}{2}(g(n-1,t)+g(n+1,t)) \tag{1}$$

for $n<k-1$,

$$g(k-1,t+1)=g(k,t)+\frac{1}{2}g(k-2,t), \qquad g(k,t+1)=\frac{1}{2}g(k-1,t),$$

and $g(n,t)=0$ for all $n>k$.

In particular (1) holds for all n not equal to $k-1$ or $k+1$. Note also that, according to the answers to the problems, we have $f(k,t)=g(k,t)$.

Consider the function $h(n,t)=2q\cdot f(n,t)+(p-q)\cdot g(n,t)$. Since $2q+p-q=p+q=1$, we have $h(0,0)=1$ and $h(n,0)=0$ for $n\neq 0$. We also have $h(k,t)=f(k,t)=g(k,t)$ for all t.

If $n\neq k-1$ and $n \neq k+1$, then

$$\begin{aligned} &h(n,t+1)=2q\cdot f(n,t+1)+(p-q)\cdot g(n,t+1)= \\ &2q\cdot\frac{1}{2}(f(n-1,t)+f(n+1,t))+(p-q)\cdot\frac{1}{2}(g(n-1,t)+g(n+1,t))= \\ &\frac{1}{2}(h(n-1,t)+h(n+1,t)). \end{aligned}$$

For $n=k-1$, we have

$$\begin{aligned} &h(k-1,t+1)=2q\cdot f(k-1,t+1)+(p-q)\cdot g(k-1,t+1)= \\ &2q\cdot\frac{1}{2}(f(k-2,t)+f(k,t))+(p-q)\cdot\left(\frac{1}{2}g(k-2,t)+g(k,t)\right)= \\ &\frac{1}{2}(2q\cdot f(k-2,t)+(p-q)\cdot g(k-2,t))+q\cdot f(k,t)+(p-q)\cdot g(k,t)= \\ &\frac{1}{2}h(k-2,t)+p\cdot h(k,t). \end{aligned}$$

For $n = k + 1$, we have

$$h(k+1, t+1) = 2q \cdot f(k+1, t+1) + (p-q) \cdot g(k+1, t+1) =$$
$$2q \cdot \frac{1}{2}(f(k,t) + f(k+2,t)) =$$
$$qf(k,t) + \frac{1}{2}(2q \cdot f(k+2,t) + (p-q) \cdot g(k+2,t)) =$$
$$qh(k,t) + \frac{1}{2}h(k+2,t),$$

which agrees with the conditions of the problem, so the answer is

$$\begin{cases} 2^{-t}\binom{t}{(n+t)/2} + (p-q) \cdot 2^{-t}\binom{t}{(n+t)/2-k} & \text{if } (n+t)/2 \in \mathbb{Z}, \text{ and } n < k, \\ 2^{-t}\binom{t}{(k+t)/2} & \text{if } (n+t)/2 \in \mathbb{Z} \text{ and } n = k, \\ 2q \cdot 2^{-t}\binom{t}{(n+t)/2} & \text{if } (n+t)/2 \in \mathbb{Z} \text{ and } n > k, \\ 0 & \text{otherwise.} \end{cases}$$

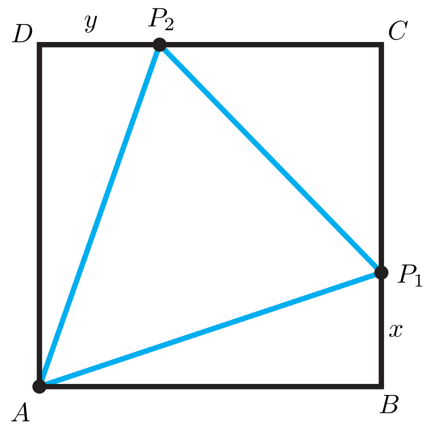
Problem 6. For $n = 2$ the answer is obviously $\sqrt{2}$ with the two points

in the opposite corners of the square.

For $n = 4$, the set of vertices of the square is a configuration S with $sd(S) = 1$. Suppose that there is a configuration with $sd(S) > 1$. Then the distance between any two vertices is more than 1. If a triangle has two sides of length > 1 and angle between them $\geq 90°$, then by Theorem of Cosines, the third side has length greater than $\sqrt{2}$. Consequently, every triangle with vertices a subset of S is acute. But it is clearly impossible. If the points of S are vertices of a convex quadrilateral, then their sum is $360°$, so one of them has at least $90°$. If they form a triangle with a point inside it, then the sum of angles formed by the interior point and two points of the triangle is also $360°$, so one of them is at least $120°$. Consequently, $d_4 = 1$.

For $n = 5$, we have a configuration with $sd(S) = \sqrt{2}/2$: take four vertices of the square and the center. Let us show that it is optimal. Divide the square into four squares with side $1/2$ in the usual way. Since we have 5 points, there will exist two vertices belonging to one square (or its boundary). But distance between any two points of a square with side $1/2$ is at most $\sqrt{2}/2$. Hence, in any configuration of five points there will be two points on distance at most $\sqrt{2}/2$ from each other.

Problem 7. Let us prove that $d_3 = \sqrt{2}(\sqrt{3}-1)$. A configuration with this value is shown on the following figure for $x = y = 2 - \sqrt{3}$.

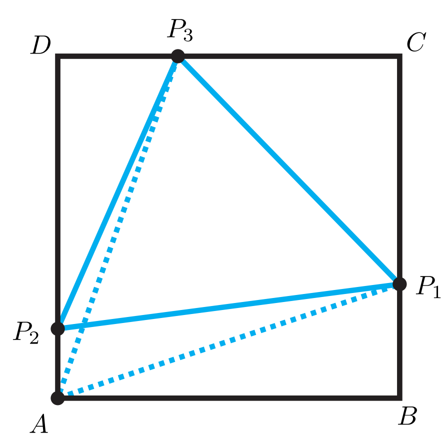


We can check that then $AP_1 = AP_2$ are equal to $\sqrt{1+x^2} = \sqrt{1+(2-\sqrt{3})^2} = \sqrt{1+4-4\sqrt{3}+3} = \sqrt{8-4\sqrt{3}} = \sqrt{2}\sqrt{4-2\sqrt{3}} = \sqrt{2}\sqrt{3-2\sqrt{3}+1} = \sqrt{2}(\sqrt{3}-1)$. The length of P_1P_2 is then $(1-x)\sqrt{2} = (1-2+\sqrt{3})\sqrt{2} = \sqrt{2}(\sqrt{3}-1)$. So AP_1P_2 is an equilateral triangle with sides of length $\sqrt{2}(\sqrt{3}-1)$.

Consider a triangle $\triangle XYZ$. Let XH be its height (i.e., XH is perpedicular to YZ). It follows from the Pythagoras Theorem, that if we move X to a point on XH outside of $\triangle XYZ$ (and on the same side of YZ as X) then two sides of $\triangle XYZ$ will become longer, and one side will remain the same.

It follows that in an optimal configuration S consisting of three points, all points must be on the boundary of the square, since otherwise we can increase the lengths of the sides.

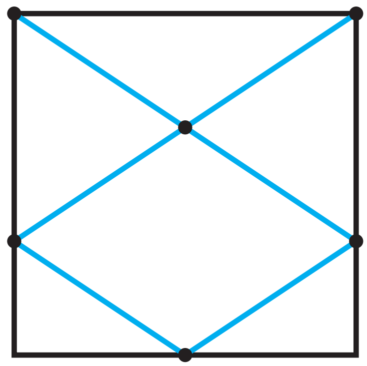
If none of the points of S is a vertex of the square, then there is a side AB of the square not containing points of S. It will be a side of a trapezoid ABP_1P_2 for P_1, P_2 from S, with two right angles P_2AB and P_1BA, see the figure below. One of the angles AP_2P_1 or P_2P_1B is right or obtuse, since their sum is $180°$. Suppose that it is AP_2P_1. Then replacing P_2 by A, we will increase two sides of the triangle formed by the points of S and not change the third one. Consequently, S can not be an optimal configuration in this case.

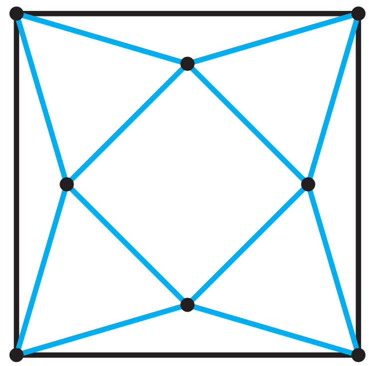


Consequently, one of the points of an optimal configuration S is a vertex of the square. Let A be this vertex of the square, as on the figure above. Suppose that all distances in S are greater than $\sqrt{2}(\sqrt{3}-1)$. Then the other two points of S can not be on the sides of the square adjacent to A. Consequently, the configuration is such as on the figure above. Then we have $\sqrt{1+x^2} > \sqrt{2}(\sqrt{3}-1)$, hence $1+x^2 > 2(4-2\sqrt{3})$, so $x^2 > 7-4\sqrt{3} = 4-4\sqrt{3}+3 = (2-\sqrt{3})^2$, so $x > 2-\sqrt{3}$. By the same argument, $y > 2-\sqrt{3}$. But then $P_1P_2 = \sqrt{(1-x)^2+(1-y)^2} < \sqrt{2}(1-(2-\sqrt{3})) = \sqrt{2}(\sqrt{3}-1)$, which is a contradiction.

Problem 8.

The optimal configurations are shown on the following figure, where the blue segments show the minimal distances.





Let us compute $x = d_n$ in these cases.

Consider the case $n = 6$. Let $2x$ be the length of the longer segment into which the vertical sides of the square are subdivided by points of S. Then $sd(S) = \sqrt{1/4+x^2}$ and $sd(S) = \sqrt{1/4+(1-2x)^2}$. It follows that $x^2 = (1-2x)^2$, hence $3x^2-4x+1 = 0$. Solving the equation (and taking into account that $x \neq 1$), we get $x = 1/3$. Consequently, we have $sd(S)$ for our configuration equal to $\sqrt{1/4+1/9} = \sqrt{13}/6$.

Consider now the case $n = 8$. Let h be the distance from one of the four internal points to the closest side. Then $sd(S) = \sqrt{1/4+h^2}$ (from the triangle formed by the point and the closest side) and $sd(S) = (1-2h)/\sqrt{2}$ (from the small square formed by the internal points). Consequently, $1/4+h^2 = (1-2h)^2/4$, which gives us quadratic equation $4h^2-8h+1 = 0$, hence $h = 1-\sqrt{3}/2$. Consequently, $sd(S) = (1-2h)/\sqrt{2} = (\sqrt{3}-1)/\sqrt{2} = (\sqrt{6}-\sqrt{2})/2$.

Problem 9. Assuming k is large enough, we are going to beat the number $sd(S_k) = 1/k$ by selecting $(k+1)^2$ points in a triangular grid. Let $\epsilon = 1/(k-1)$ and consider a triangular grid such that points $(0,0)$ and $(\epsilon, 0)$ are neighboring vertices. All vertices of that grid have coordinates of the form $(m_1\epsilon/2, m_2\epsilon\sqrt{3}/2)$, where m_1 and m_2 are arbitrary integers of the same parity (that is, either both even or both odd). By construction, the minimal distance between vertices in the grid is ϵ.

Since $\epsilon > 1/k$, we only need to show that at least $(k+1)^2$ of those vertices fit inside the square Q.

A point $(m_1\epsilon/2, m_2\epsilon\sqrt{3}/2)$ belongs to the square Q if $0 \leq m_1\epsilon/2 \leq 1$ and $0 \leq m_2\epsilon\sqrt{3}/2 \leq 1$. Equivalently, if $0 \leq m_1 \leq 2(k-1)$ and $0 \leq m_2 \leq 2(k-1)/\sqrt{3}$. There are k possibilities for m_1 to be an even integer and $k-1$ possibilities to be an odd integer. Further, let $k_1 = \lfloor (k-1)/\sqrt{3} \rfloor$. Then $2k_1 \leq 2(k-1)/\sqrt{3}$. Hence there are at least $k_1 + 1$ possibilities for m_2 to be an even integer and at least k_1 possibilities to be an odd integer.

By the above the number of vertices of the triangular grid that fit inside the square Q is at least $k(k_1+1) + (k-1)k_1$. Note that $k_1 > ((k-1)/\sqrt{3}) - 1$. Therefore

$$
\begin{aligned}
k(k_1+1) + (k-1)k_1 > k\frac{k-1}{\sqrt{3}} + (k-1)\left(\frac{k-1}{\sqrt{3}} - 1\right) = \\
\frac{2}{\sqrt{3}}k^2 - (1+\sqrt{3})k + \left(1 + \frac{1}{\sqrt{3}}\right) > \\
\frac{2}{\sqrt{3}}k^2 - (2\sqrt{3})k + \frac{2}{\sqrt{3}} = \\
\frac{2}{\sqrt{3}}(k+1)^2 - \frac{10}{\sqrt{3}}k = \frac{2}{\sqrt{3}}(k+1)^2\left(1 - \frac{5k}{(k+1)^2}\right).
\end{aligned}
$$

Since $2/\sqrt{3} > 1$ and $5k/(k+1)^2 \to 0$ as $k \to \infty$, it follows that $k(k_1+1) + (k-1)k_1 \geq (k+1)^2$ if k is sufficiently large.