2022 Power Team
Texas A&M High School Students Contest
November 2022

Placing points into shapes, cutting pies, and escaping shapes

1. A set of points is located on the plane so that any triangle with vertices at these points has area at most 1. Prove that all of these points lie in a triangle of area 4 (when we say "lie in a triangle" we mean that they may also lie on its boundary).

2. (a) Is it possible to choose 6 points in the disc of radius 1 such that the distance between any two of them is greater than 1? Prove your answer.

   (b) Is it possible to choose 441 points in the disc of radius 10 such that the distance between any two of them is greater than 1? Prove your answer.

3. A pie has a shape of a regular n-gon inscribed into a circle of radius 1 (we assume that the pie is planar, ignoring its thickness). From the middle of each edge of this n-gon one makes a cut in an arbitrary direction along a segment of length at least 1. Prove that after these cuts a piece is cut off from the pie.

4. (a) A pie has a shape of a square of side length 1 (again we assume that the pie is planar, ignoring its thickness). Divide each edge of this pie into 3 equal pieces by marking two corresponding points on each edge. Cut from the square four corner triangles which are obtained as follows: for every such triangle one vertex coincides with a vertex of the pie and two other vertices are the marked points on the adjacent edges closest to this vertex. Then repeat the process with the resulting octagonal pie: divide each edge of this octagon into 3 equal pieces by marking two corresponding points on each edge, then cut from the octagon eight triangles which are obtained as follows: for every such triangle one vertex coincides with a vertex of the octagon and two other vertices are the marked points on the adjacent edges closest to this vertex. Find the area of the pie obtained by repeating this procedure infinitely many times.

   (b) Solve the same problem if on each step the edge is divided into p equal pieces with $p > 2$ (as in the previous item for any triangle removed on the nth step one vertex coincides with a vertex of the polygon of the previous $(n-1)$st step and two other vertices are the marked points on the adjacent edges closest to this vertex).

5. A hare sits in the center of a square, and one wolf sits in each of the four corners. Wolves can only run along the borders of the square, and the hare can move in any direction. At each moment the wolves and the hare know the location of all participants and there is no delay in the reaction of any participants to the change of location of any other.

(a) Can the hare run out of the square without being caught by at least one wolf if the maximum speed of the wolves is 1.4 times greater than the maximum speed of the hare? Prove your answer.

(b) At what minimum ratio r^* of the maximum speeds of the wolves and the hare do wolves always have a strategy with which they can catch the hare? Prove your answer.

6. Assume that in the setting of the previous problem a moose tries to escape the square instead of a hare. A moose is strong enough so that one wolf cannot hold him, but two wolves can.

   (a) At what minimum ratio r^* of the maximum speeds of the wolves and the moose do wolves always have a strategy with which they can catch the moose? Here, as in Problem 5, we assume that the moose starts at the center of the square and the wolves start at the corners. Prove your answer.

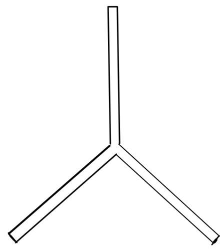
   (b) Assuming that the ratio of the maximum speeds of the wolves and the moose is $r > r^*$, where r^* is the critical ratio from part (a), find the set of all initial positions of the moose inside the square such that the wolves always have a strategy with which they can catch the moose (if they start at the corners).

7. An island in a sea is so small that its size can be ignored and it can be assumed to be a point. A lighthouse is installed on the island. At every time moment the beam of the projector can illuminate a narrow sector of the sea surface of length L (the thickness of the sector is ignored). The projector rotates uniformly around the vertical axis (in a fixed direction), making one revolution per time interval T. A boat, which can move at a speed v, must approach the island imperceptibly (that is, without falling into the searchlight beam).

   (a) Can the boat succeed if its maximal velocity v is $0.9\frac{L}{T}$?

   (b) Find the minimal v_0 such that if $v > v_0$ the boat can succeed? Your answer can be represented as a solution of an equation involving trigonometric functions (without trying to solve the equation).

   (c) Assume that in the setting of the previous problem two lighthouses are installed on the island and they rotate in the same direction uniformly around the vertical axis, making one revolution per time interval T, illuminate narrow sectors of length L (the thickness of the sector is ignored), and have angle $\theta \in (0, \pi]$ between them. Answer the same question as in the previous item.



8. Three narrow rectangular hallways of length ℓ meet in a common point as shown in the figure above. One can ignore the width of the hallways so each of them is considered as a line segment. The hallways contain no doors. A cop and a gangster run along these hallways so that the maximal speed of the cop is twice the maximal speed of the gangster. A cop sees the gangster if they are both in the same hallway and the distance between them is not greater than r. This includes the case when one of them is in the center and when the cop is in the center he can watch simultaneously along all three hallways. Prove that under each of the following conditions on r, the cop can catch the gangster from any starting position:

   (a) $r \geq \frac{\ell}{3}$;

   (b) $r \geq \frac{\ell}{4}$;

   (c) $r > \frac{\ell}{5}$;

   (d) $r > \frac{\ell}{7}$.