

**Algebra Qualifying Examination**  
**August 14, 2009**

**Instructions:**

- There are nine Algebra questions worth a total of 100 points. Individual point values are listed next to each problem.
- Credit awarded for your answers will be based on the correctness of your answers as well as the clarity and main steps of your reasoning. “Rough working” will not be accepted: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but you will not earn points for using it as a step in your reasoning).
- All rings have an identity and all modules are unital.

**Notation:** Throughout,  $\mathbb{Z}$  denotes the integers,  $\mathbb{Q}$  denotes the rational numbers,  $\mathbb{R}$  denotes the real numbers, and  $\mathbb{C}$  denotes the complex numbers.

1. (5 points) Prove that there are no simple groups of order 124.
2. (10 points) Let  $G$  be a group of order  $1995=3(5)(7)(19)$ . Show that  $G$  has a normal cyclic subgroup of index 3.
3. (10 points) Find the Jordan Canonical Form  $J$  and matrix  $P$  such that  $PAP^{-1} = J$  where

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 3 \end{pmatrix}.$$

4. (15 points)
  - (a) Describe all maximal ideals in  $\mathbb{C}[x]$ .
  - (b) Describe all maximal ideals in  $\mathbb{R}[x]$ .
  - (c) Describe (up to isomorphism) the fields that may be obtained as quotients  $\mathbb{R}[x]/M$ , where  $M$  is a maximal ideal of  $\mathbb{R}[x]$ .
5. (10 points) Let  $K$  be a field and let  $K^*$  be the non-zero elements in  $K$ . A *discrete valuation* on  $K$  is a function  $\nu : K^* \rightarrow \mathbb{Z}$  such that
  - i.  $\nu(ab) = \nu(a) + \nu(b)$  for all  $a, b \in K^*$ , i.e.,  $\nu$  is a homomorphism from the multiplicative group of the field to the integers.
  - ii.  $\nu$  is surjective.
  - iii.  $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$  for all  $a, b \in K^*$  with  $a + b \neq 0$ .The set  $R = \{x \in K^* | \nu(x) \geq 0\} \cup \{0\}$  is call the *valuation ring* of  $\nu$ .
  - (a) Prove that  $R$  is a subring of  $K$  which contains the identity.
  - (b) Prove that, for each nonzero  $x \in K$ , either  $x$  or  $x^{-1}$  is in  $R$ .

6. (15 points) Let  $R$  be a ring with  $1 \neq 0$  and  $M$  an  $R$ -module. Let
- $$T(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\},$$
- $$A(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}.$$
- (a) Prove that  $A(M)$  is a left ideal of  $R$ .
- (b) If  $R$  is *commutative*, prove that  $T(M)$  is an  $R$ -submodule of  $M$ .
- (c) Give an example, with explanation, of a ring  $R$  and an  $R$ -module  $M$  such that  $T(M)$  is *not* an  $R$ -submodule of  $M$ . (Hint: consider the  $R$ -module  $R$ , under left multiplication.)
7. (10 points) Let  $K \subseteq F \subseteq L$  be a tower of fields. Let  $\theta \in L$  be algebraic over  $K$  and let  $p(x)$  be the minimal polynomial of  $\theta$  over  $K$ . Prove that  $F \otimes_K K[\theta] \cong F[x]/(p(x))$  as  $F$ -algebras.
8. (15 points)
- (a) Exhibit a finite field  $K$  consisting of 8 elements. (That is, describe or construct such a field explicitly.)
- (b) What is the prime subfield  $F$  of  $K$ ?
- (c) Is  $K$  a Galois extension of  $F$ ? Describe the group  $\text{Aut}(K/F)$ .
9. (10 points) Let  $f(x) \in \mathbb{Q}[x]$  be a polynomial of degree 3, and  $G$  its Galois group. Prove that if  $G$  is the cyclic group of order 3, then  $f(x)$  splits completely over  $\mathbb{R}$ .