Algebra Qualifying Examination
11 August 2011

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed by each problem.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. "Rough working" will not receive credit: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but may not be used as a step in your reasoning).
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, check with the exam administrator. Do not interpret the problem in a way that makes it trivial.

Notation: Throughout, $\mathbb{Q}$ and $\mathbb{C}$ denote the field of rational or complex numbers, respectively. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{F}$ denotes the field with two elements.

1. (12 points) Show that any group of order 455 is cyclic.

2. (13 points) Decompose 35 into product of prime elements of $\mathbb{Z}[i]$ (and show this is indeed a prime decomposition).

3. (11 points) Suppose that $[\mathbb{Q}(u) : \mathbb{Q}]$ is odd. Show that $\mathbb{Q}(u^2) = \mathbb{Q}(u)$.

4. (11 points) Find an inverse of $(1 + x)^3$ in $\mathbb{F}_2[[x]]$.

5. (13 points) Let M be a module over a ring R, N and P submodules in M. Define $(N : P) = \{r \in R | rP < N\}$. Show that $(N : P)$ is an ideal of R. Show also that $(N : P) = Ann((N + P)/N)$, where if L is an R module then $Ann(L) = \{r \in R | rL = 0\}$.

6. (12 points) Let R be a commutative ring with identity, and let P and Q be projective R-modules. Prove that $P \bigotimes_R Q$ is a projective R-module.

7. (14 points) Let $A \subset End(V)$ be a subring of linear operators on an N-dimensional $\mathbb{C}$-vector space V and define $tr : A \to \mathbb{C}$ by $tr(a) = \sum_{i=1}^{N} \lambda_i$ where $\lambda_1, \ldots, \lambda_N$ are the roots of the characteristic polynomial of a (i.e. the eigenvalues with multiplicities).
   (a) Show that $Ann(tr) := \{s \in A | tr(sb) = 0, \text{ for all } b \in A\}$ is a 2-sided ideal in A.
   (b) Use the fact that if $tr(s^k) = 0$ for all $k \geq 1$ then s is nilpotent to prove that every element of $Ann(tr)$ is nilpotent. Conclude that $Ann(tr)$ is contained in every maximal left ideal of A.

8. (14 points) Let θ be a root of $x^3 - 3x + 1$. Prove that the splitting field of this polynomial is $\mathbb{Q}(\theta)$ and find the Galois group over $\mathbb{Q}$. Show that the other roots of this polynomial can be written in the form $a + b\theta + c\theta^2$ for some $a, b, c \in \mathbb{Q}$. Determine the other roots explicitly in terms of θ.