Algebra Qualifying Examination
7 August 2012

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed next to each problem number.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and must be legible.
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, check with the exam administrator. Do not interpret the problem in a way that makes it trivial.
- Start each problem on a new page.

Notation: Throughout, $\mathbb{Q}$ and $\mathbb{C}$ denote the field of rational and complex numbers, respectively. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{F}_q$ denotes the field with q elements, where q is a power of a prime number.

1. (10 points) Let N be a normal subgroup of a finite group G. Suppose $|N| = 5$ and that $|G|$ is odd. Prove N is contained in $Z(G)$, the center of G.

2. (15 points) Show that no group of order p^2q is simple where p and q are distinct primes. (Hint: Consider two cases $q < p$ and $p < q$. You may assume known that no group of order 12 is simple.)

3. (13 points) Let R be a commutative ring with $1_R \neq 0_R$ and let $M \neq 0$ be a simple R-module, i.e. M has no non-zero proper submodules.

   a) Show that M is isomorphic as R-module to R/m for some maximal ideal $m \subseteq R$.
   b) Show that if M_1, M_2 are both simple R-modules and $\varphi : M_1 \to M_2$ is an R-module homomorphism then φ is either the zero map or an isomorphism.

4. (12 points) Let R be a commutative ring with $1_R \neq 0_R$ and $0 \to M \xrightarrow{f} N$ an exact sequence of R-modules, i.e. f is injective. Let P be a projective R-module. Show that $M \otimes_R P \xrightarrow{f\otimes 1} N \otimes_R P$ is also injective. (You may use the fact that tensor product commutes with direct sum, that is,

$$\left(\bigoplus_{i\in I} A_i\right) \otimes_R B \cong \bigoplus_{i\in I}(A_i \otimes_R B)$$

in the obvious way.)

5. (13 points) Consider the polynomial $f(X) = X^4 - 2$ over the rational numbers $\mathbb{Q}$.

   1) Show $f(X)$ is irreducible over $\mathbb{Q}$.

   2) What is the Galois group of the splitting field K of $f(X)$ over $\mathbb{Q}$?

   3) Construct two specific automorphisms of K over $\mathbb{Q}$ that generate Gal $(K/\mathbb{Q})$. Hint: Consider the intermediate fields $\mathbb{Q}(i)$ and $\mathbb{Q}(\alpha)$ where α is the real fourth root of 2.

6. (12 points) Let $<,>: V \times V \to \mathbb{R}$ be a symmetric bilinear form on a finite dimensional real vector space V. Thus for $v_1, v_2, v, w \in V$ and $\alpha \in \mathbb{R}$, we have

$$< v_1 + v_2, v > = < v_1, v > + < v_2, v >$$
$$< \alpha v, w > = \alpha < v, w >$$
$$< v, w > = < w, v > .$$

   Show there exists an orthogonal basis $\{v_1, ..., v_n\}$ for V, i.e. $< v_i, v_j >= 0$ for $i \neq j$, with $< v_i, v_i >= 1, -1$, or 0 for every i.

7. (15 points) Construct the finite field $\mathbb{F}_9$ with 9 elements and find a generator for the multiplicative group $\mathbb{F}_9^{\times}$.

8. (10 points) Let D be a unique factorization domain and suppose $\pi \in D$ is irreducible.

   a) Show $P = (\pi)$ is a prime ideal.

   b) Let $S = D - P$. Note $1 \in S$ and S is closed under multiplication. Show the ring $S^{-1}D$ is a principal ideal domain.
   Note: $D \subseteq S^{-1}D \subseteq K$, the field of fractions of the domain D.