**Algebra Qualifying Examination**
**August 13, 2013**

**Instructions:**

- There are 8 problems worth a total of 100 points. Individual point values are listed next to each problem number.
- Credit awarded for your answers will be based on both the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and must be legible.
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, please check with the exam administrator. Do not interpret a problem in a way that makes it trivial.
- Start each problem on a new page.

**Notation:** Throughout, let $\mathbb{Z}$ denote the ring of integers; let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers respectively; let $\mathbb{F}_q$ denote the finite field with $q$ elements, where $q$ is a power of a prime number. For a ring $R$ with 1, we let $R^\times$ denote its unit group. For a Galois extension of fields $L/K$, let $\mathrm{Gal}(L/K)$ denote its Galois group.

1. [10 points] Let $R$ be a commutative ring with $1 \neq 0$, and suppose that for every $r \in R$ there is some $n > 1$ (depending on $r$) so that $r^n = r$. Prove that every prime ideal of $R$ is maximal.

2. [12 points] Let $G$ be a group of order $132 = 11 \cdot 12$. Show that $G$ has a normal subgroup of order 11 or a normal subgroup of order 12. (Hint: In the case that $G$ does not have a normal subgroup of order 11, show that for an element $x \in G$ of order not 1 or 11, the centralizer $C_G(x)$ of $x$ has order 12.)

3. [10 points] Let $g = x^2 + 2x - 1 \in \mathbb{F}_5[x]$.

   (a) Let $E$ be the quotient ring $\mathbb{F}_5[x]/(g)$. Show that $E$ is a field. What is $|E|$ and why?

   (b) Let $\alpha$ denote $x + (g) \in E$. What is the order of $\alpha$ in $E^\times$ and why?

4. [10 points] Let $R$ be a unique factorization domain.

   (a) Show that $\pi$ is irreducible in $R$ if and only if the ideal $(\pi)$ in $R$ is a prime ideal.

   (b) Let $\pi \in R$ be irreducible, and suppose that $Q \subseteq (\pi)$ is a non-zero prime ideal. Show that $Q = (\pi)$. (Hint: Show that $Q$ can be generated by irreducible elements.)

5. [12 points] Let $R$ be a commutative ring with $1 \neq 0$, and let $I$ and $J$ be ideals of $R$. Show that there is an $R$-module isomorphism $\phi : R/I \otimes_R R/J \to R/(I + J)$ with $\phi(\overline{x} \otimes \overline{y}) = \overline{xy}$. (Here $\overline{x}$ denotes the coset $x + I \in R/I$, and similarly $\overline{y} = y + J \in R/J$ and $\overline{xy} = xy + (I + J) \in R/(I + J)$.)

6. [20 points: (a) 5 pts.; (b) & (c) 4 pts. each; (d) 7 pts.] Let $p$ be an odd prime number, and let

$$f(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + \cdots + x + 1 \in \mathbb{Z}[x].$$

(a) Show that $f$ is irreducible in $\mathbb{Q}[x]$ using Eisenstein's criterion.

(b) Let $\zeta = e^{2\pi i/p} \in \mathbb{C}$, and let $K = \mathbb{Q}(\zeta)$. Show that $K$ is the splitting field of $f$ over $\mathbb{Q}$.

(c) Let $G = \mathrm{Gal}(K/\mathbb{Q})$. For $\sigma \in G$, show that there is a unique integer $m(\sigma) \in \{1, \ldots, p-1\}$ such that

$$\sigma(\zeta) = \zeta^{m(\sigma)}.$$

(d) Prove that the function $m : G \to (\mathbb{Z}/p\mathbb{Z})^\times$ defined in (c) is a group isomorphism.

7. [10 points: (a) & (b) 3 pts. each; (c) 4 pts.] Let $R$ be a ring with identity $1 \neq 0$. Let $M$ be a left $R$-module. You may assume that all $R$-modules under consideration are unitary (i.e., that $1 \cdot m = m$ for all $m \in M$).

(a) Complete the following sentence to define a projective module: "$M$ is a projective left $R$-module if and only if given a __________ $R$-module homomorphism $g : L \to N$ and any $R$-module homomorphism $f : M \to N$, we have ________________________."

(b) The definition in (a) is equivalent to a number of other statements about $M$. Provide one of them.

(c) Show that if $M_i$, $i \in I$, are projective left $R$-modules, then the direct sum $\bigoplus_{i \in I} M_i$ is a projective left $R$-module.

8. [16 points] Let $G$ be a group, and let $V$ be a 2-dimensional vector space over a field $K$. Suppose we have an action of $G$ on $V$, $(g, v) \mapsto g \cdot v$, defined in such a way that for all $g \in G$, $c \in K$, and $v$, $w \in V$,

$$g \cdot (cv) = c(g \cdot v),$$
$$g \cdot (v + w) = g \cdot v + g \cdot w.$$

(a) Use the action of $G$ to define a group homomorphism $\rho : G \to \mathrm{GL}_2(K)$, where $\mathrm{GL}_2(K)$ is the group of invertible $2 \times 2$ matrices with entries in $K$.

(b) Suppose there exist functions $\beta : G \to K$ and $\delta : G \to K^\times$ so that for all $g \in G$,

$$\rho(g) = \begin{pmatrix} 1 & \beta(g) \\ 0 & \delta(g) \end{pmatrix}.$$

Show that $V$ has a 1-dimensional subspace $W$ that is fixed by $G$.

(c) Show that $\delta$ from (b) is a group homomorphism and that $\beta(g_1 g_2) = \beta(g_2) + \beta(g_1)\delta(g_2)$ for all $g_1$, $g_2 \in G$.

(d) Fix $v = \begin{pmatrix} a \\ b \end{pmatrix} \in K^2$, with $b \neq 0$, and let $U = Kv$ be the 1-dimensional space spanned by $v$. Suppose further that for all $g \in G$,

$$\rho(g)(U) \subseteq U.$$

Show that there is some $c_0 \in K$ so that for all $g \in G$ we have $\beta(g) = \delta(g)c_0 - c_0$. Check that $\beta$ in this form satisfies the condition in (c).