Algebra Qualifying Examination
August 5, 2024

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed next to each problem number.
- Credit awarded for your answers will be based on the correctness of your answers, as well as the clarity and main steps of your reasoning. "Rough working" will not receive credit: answers must be legible and written in a structured and understandable manner. Do scratch work on a separate page.
- Read all problems first. Make sure that you understand them, and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- You may use a calculator to check your computations (but it may not be used as a step in your reasoning).

Notation: Throughout, let $\mathbb{Z}$ denote the ring of integers; let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers respectively. For a set S, we let $|S|$ denote its cardinality. All non-zero rings are assumed to have a multiplicative identity $1 \neq 0$. For a Galois extension of fields L/K, let $\mathrm{Gal}(L/K)$ denote its Galois group.

1. [12 points] Let G be a finite group, and let $N \subseteq G$ be a normal subgroup.
   (a) Let $\mathrm{Aut}(N)$ denote the automorphism group of N. Show that there is a group homomorphism $\phi : G \to \mathrm{Aut}(N)$ whose kernel is the centralizer of N in G.
   (b) Suppose $|N| = p$, where p is the smallest prime divisor of $|G|$. Prove that N is contained in the center of G.

2. [12 points] Let G be a group of order $5 \cdot 13 \cdot 23 \cdot 43$. How many elements of order 5 are contained in G?

3. [12 points] Let R be a commutative ring that satisfies the *descending chain condition*. That is, for any descending chain of ideals, $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$, there must exist $t \geq 1$ so that $I_k = I_t$ for $k \geq t$.
   (a) Let $x \in R$. Show that there exists $t \geq 1$ and $r \in R$ so that $x^t = rx^{t+1}$.
   (b) For a prime ideal P of R, use part (a) to prove that R/P is a field.

4. [12 points] We make $\mathbb{C}^3$ into a $\mathbb{C}[x]$-module by setting $f(x) \cdot v = f(A)v$, where

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 5 \end{pmatrix}.$$

   Find polynomials $p_i(x) \in \mathbb{C}[x]$ and exponents e_i such that $\mathbb{C}^3 \cong \bigoplus_i \mathbb{C}[x]/(p_i(x)^{e_i})$ as $\mathbb{C}[x]$-modules. Justify your answer.

5. [12 points] Let R be a ring with $1 \neq 0$. Let M be a left R-module, and $N \subseteq M$ a submodule. Suppose that M and M/N are projective R-modules. Prove that N is also projective.

6. [14 points] Let K be a field, and let $R = K[x, y]$ be the polynomial ring in variables x and y over K. Let $M = (x, y) \subseteq R$ (i.e., $M = Rx + Ry$, the ideal generated by x and y).

   (a) We make K into an R-module by setting $f(x, y) \cdot \alpha = f(0, 0)\alpha$, for $f \in R$ and $\alpha \in K$. Show that there is an R-module homomorphism $\phi : M \otimes_R M \to K$ such that on pure tensors
   $$\phi(f \otimes g) = \frac{\partial f}{\partial x}(0, 0) \cdot \frac{\partial g}{\partial y}(0, 0).$$
   Here partial derivatives are defined formally in the usual way.

   (b) Show that $x \otimes y \neq y \otimes x$ in $M \otimes_R M$.

   (c) Show that $x \otimes y - y \otimes x$ is non-zero and torsion in $M \otimes_R M$. That is, there exists $r \in R$, $r \neq 0$, such that $r(x \otimes y - y \otimes x) = 0$.

7. [14 points] Let $f = x^4 - 2 \in \mathbb{Q}[x]$, and let $E \subseteq \mathbb{C}$ be the splitting field of f.

   (a) Show that $E = \mathbb{Q}(\sqrt[4]{2}, i)$ and determine $[E : \mathbb{Q}]$.

   (b) Show that there is an automorphism $\sigma : E \to E$ such that $\sigma(\sqrt[4]{2}) = i\sqrt[4]{2}$ and $\sigma(i) = i$.

   (c) Let $\tau : E \to E$ be the restriction of complex conjugation. As elements of $\mathrm{Gal}(E/\mathbb{Q})$, what are the orders of τ, σ, and $\sigma^2\tau$?

   (d) Let $H = \langle \sigma^2\tau \rangle \subseteq \mathrm{Gal}(E/\mathbb{Q})$. What is the fixed field of H?

8. [12 points] Fix a field K, and let $K[t]$ be the polynomial ring in a variable t over K. Let $K(t)$ denote the field of fractions of $K[t]$. That is, $K(t)$ is the field of rational functions over K. In a natural way, $K \subseteq K(t)$ as a subfield. Let $\sigma : K(t) \to K(t)$ be defined by setting for polynomials $f, g \in K[t]$, $g \neq 0$,
   $$\sigma\left(\frac{f(t)}{g(t)}\right) = \frac{f(t+1)}{g(t+1)}.$$

   (a) Show that σ is a well-defined field automorphism of $K(t)$ that restricts to the identity on K.

   (b) Depending on the characteristic of K, what is the order of σ as an element of the group $\mathrm{Aut}(K(t)/K)$?

   (c) Let $K = \mathbb{F}_2$, the field with 2 elements. Let $G = \langle \sigma \rangle \subseteq \mathrm{Aut}(\mathbb{F}_2(t)/\mathbb{F}_2)$, and let $E \subseteq \mathbb{F}_2(t)$ be the fixed field of G. Show that $E = \mathbb{F}_2(t^2 + t)$. (Hint: What is $[\mathbb{F}_2(t) : \mathbb{F}_2(t^2 + t)]$?)