

**Algebra Qualifying Examination**  
**10 January 2011**

**Instructions:**

- There are 8 problems worth a total of 100 points. Individual point values are listed by each problem.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. “Rough working” will not receive credit: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but may not be used as a step in your reasoning).
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, check with the exam administrator. Do not interpret the problem in a way that makes it trivial.

1. (13 points)

- Show that the set theoretic union of two subgroups cannot be a subgroup, unless one of the subgroups is contained in the other.
- Show that every vector space over  $GF(2)$  of finite dimension  $d > 1$  can be written as the set theoretic union of three of its proper subspaces.

2. (14 points)

Let  $A$  be a free Abelian group of finite rank.

- (a) Let  $B$  be a subgroup of  $A$  such that  $A = B + pA$  for some prime number  $p$ . Prove that  $B$  is a subgroup of finite index in  $A$ .
- (b) Let  $B$  be a subgroup of  $A$  such that  $A = B + pA$  for any prime number  $p$ . Prove that  $A = B$ .

3. (11 points) Prove that a group of order 616 is solvable.

4. (11 points) Suppose  $T$  is a linear operator on an  $n$ -dimensional vector space  $V$  over a field  $F$  such that for any non-zero  $v$  in  $V$  the set  $\{T^i(v) : i = 0, \dots, n - 1\}$  is linearly independent. Show that the characteristic polynomial of  $T$  is irreducible over  $F$ .

5. (16 points)

- (a) Let  $F$  be a field and let  $R = F[t]$  be the polynomial ring. Let  $R'$  be the ring extension  $R[x]/(tx - 1)$  obtained by adjoining an inverse of  $t$  to  $R$ . Prove that this ring can be identified as the ring  $F[x, x^{-1}]$  of Laurent polynomials which are finite linear combinations of powers of  $x$ , negative exponents included.
- (b) Let  $F$  as before be a field. Prove that the ring  $F[x, x^{-1}]$  of Laurent polynomials is a principal ideal domain.

6. (6 points) Show that the group of units in  $\mathbb{Z}[(1 + \sqrt{5})/2]$  is infinite.
7. (10 points) An  $R$ -module  $M$  is called faithful if  $rM = 0$  for  $r \in R$  implies  $r = 0$ . Let  $M$  be a finitely generated faithful  $R$ -module and let  $I$  be an ideal of  $R$  such that  $IM = M$ . Prove  $I = R$ .
8. (19 points) Let  $\alpha = \sqrt[3]{2}$ ,  $\beta = \sqrt{3}$ , and  $\gamma = \alpha + \beta$ . Let  $L$  be the field  $\mathbb{Q}(\alpha, \beta)$ , and let  $K$  be the splitting field of the polynomial  $(x^3 - 2)(x^2 - 3)$  over  $\mathbb{Q}$ .
- (a) Determine the irreducible polynomial  $f$  for  $\gamma$  over  $\mathbb{Q}$ , and its roots in  $\mathbb{C}$ .
  - (b) Determine the degree  $[K : L]$ .
  - (c) Determine the Galois group of  $K/\mathbb{Q}$ .