Algebra Qualifying Examination
January 7, 2013

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed next to each problem number.
- Credit awarded for your answers will be based on both the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and must be legible.
- You may use a calculator to check or perform simple computations, but it may not be used as a step in your reasoning.
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, please check with the exam administrator. Do not interpret a problem in a way that makes it trivial.
- Start each problem on a new page.

Notation: Throughout, let $\mathbb{Z}$ denote the ring of integers; let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers respectively; let $\mathbb{F}_q$ denote the finite field with q elements, where q is a power of a prime number. For a Galois extension of fields L/K, let $\mathrm{Gal}(L/K)$ denote its Galois group.

1. [10 points] Let G be a group of order $56 = 2^3 \cdot 7$. Show that G is not simple.

2. [10 points] Let G be a group of order $200 = 2^3 \cdot 5^2$, and let S_8 be the symmetric group on $\{1, \ldots, 8\}$. Show that there exists a group homomorphism $\psi : G \to S_8$ with proper non-trivial kernel. (Hint: Find a set with 8 elements on which G acts.)

3. [15 points] Give examples of the following objects. Be sure to verify that your examples satisfy the desired properties.
   (a) An irreducible polynomial over $\mathbb{Q}$ that is irreducible by Eisenstein's criterion for $p = 5$.
   (b) A unique factorization domain that is not a principal ideal domain.
   (c) A finite extension of the rational function field $\mathbb{F}_p(x)$, for p a prime, that is normal but not separable.

4. [10 points] Let R be a commutative ring with $1 \neq 0$. Let M and N be left R-modules such that M is finitely generated and N is noetherian. Show that $M \otimes_R N$ is noetherian.

5. [10 points] Let R be a commutative ring with $1 \neq 0$, and let N be a left R-module. For a prime ideal $\mathfrak{p} \subseteq R$, let $R_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ denote their localizations at $\mathfrak{p}$. That is, $R_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}R$ and $N_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}N$, where $S_{\mathfrak{p}} = R - \mathfrak{p}$. Show that the following are equivalent:

   (i) $N = \{0\}$,

   (ii) $N_{\mathfrak{p}} = \{0\}$ for all prime ideals $\mathfrak{p} \subseteq R$,

   (iii) $N_{\mathfrak{m}} = \{0\}$ for all maximal ideals $\mathfrak{m} \subseteq R$.

   (Hint: First show that if $x \neq 0$ is an element of a module M over a commutative ring R with 1, then the set $A(x) := \{r \in R : r \cdot x = 0\}$ is an ideal of R.)

6. [15 points] Let $V = \mathbb{Q}(\sqrt{2} + \sqrt{3})$, and let $K = \mathbb{Q}(\sqrt{2})$.

   (a) Show that $V/\mathbb{Q}$ is a Galois extension and determine $\mathrm{Gal}(V/\mathbb{Q})$ up to isomorphism.

   (b) Let $T : V \to V$ be defined by $T(\alpha) = (1 + \sqrt{2})\alpha$. Verify that T is a linear transformation of V as a vector space over $\mathbb{Q}$. By choosing a basis for V as $\mathbb{Q}$-vector space, represent T as a matrix for this basis and find its characteristic polynomial.

   (c) Let $\mathrm{Id} : K \to K$ denote the identity map. Find a K-basis of $K \otimes_{\mathbb{Q}} V$ consisting of eigenvectors for the K-linear map

   $$\mathrm{Id} \otimes T : K \otimes_{\mathbb{Q}} V \to K \otimes_{\mathbb{Q}} V.$$

   Present these eigenvectors as linear combinations of pure tensors.

7. [15 points] Let $f,\, g \in \mathbb{Q}[x]$ be non-constant polynomials. Let $H \subseteq \mathbb{C}$ be the splitting field of f, let $K \subseteq \mathbb{C}$ be the splitting field of g, and let $L \subseteq \mathbb{C}$ be the splitting field of fg.

   (a) Find an injective group homomorphism

   $$\phi : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(H/\mathbb{Q}) \times \mathrm{Gal}(K/\mathbb{Q}).$$

   (b) For $(\sigma, \tau) \in \mathrm{Gal}(H/\mathbb{Q}) \times \mathrm{Gal}(K/\mathbb{Q})$, find a necessary and sufficient criterion for (σ, τ) to be in the image of ϕ.

8. [15 points] Let R be a ring with $1 \neq 0$, and let M be a finitely generated left R-module.

   (a) Suppose that M is projective as a left R-module. Then prove there exist elements $m_1, \ldots, m_k \in M$ and R-module homomorphisms $f_i : M \to R$, $1 \leq i \leq k$, such that for all $m \in M$,

   $$m = \sum_{i=1}^{k} f_i(m) m_i.$$

   (b) Prove that the converse of (a) is true.