Algebra Qualifying Examination
8 January 2018

Instructions:

- There are nine questions worth a total of 100 points. Individual point values are indicated with each problem number.
- Read all problems first; make sure that you understand them and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial. (E.g. #6.)
- Credit is awarded based both on the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and be legible. State clearly any major theorems you use (hypotheses and conclusions). Justify your reasoning.
- Start each problem on a new page, clearly marking the problem number and your name on that page. Do 'scratch work' on a separate page.
- Rings always have an identity (otherwise they are rng) and all R-modules are left modules.

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(1) [10] Consider an attempt to define an $\mathbb{R}$-linear map

$$f \,:\, \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \qquad \text{or} \qquad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}\,,$$

in either direction given by the formula

$$f(x \otimes y) = x \otimes y\,.$$

In which direction is this map well-defined? Is it then surjective? Is it injective?

(2) [10] Let R be an integral domain with field of fractions K, and let $\overline{K}$ be an algebraic closure of K. Fix $\alpha \in \overline{K}$. Suppose that $M \subseteq \overline{K}$ is a finitely generated R-submodule such that

$$\alpha M \subseteq M\,.$$

Prove that there is a monic polynomial $f \in R[x]$ such that $f(\alpha) = 0$. (Hint: If M is generated over R by $m_1, \ldots, m_n$, then consider the characteristic polynomial of an $n \times n$ matrix over R that relates the two vectors

$$\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \alpha m_1 \\ \vdots \\ \alpha m_n \end{pmatrix}$$

in $\overline{K}^n$.)

(3) [10] Prove: If a group G contains a proper subgroup of finite index, then it contains a proper normal subgroup of finite index.

(4) [10] How many elements of order 7 are there in a simple group of order 168?

(5) [10] Determine all homomorphisms from $\mathbb{Q}$ to $\mathbb{Z}$, and all homomorphisms from $\mathbb{Z}$ to $\mathbb{Q}$.

(6) [12] Give the definition for an element of a commutative ring R to be prime and the definition for an element of a commutative ring R to be irreducible. Prove that in a principal ideal domain every irreducible element is prime.

(7) [10] Let R be a commutative ring, and let M be a noetherian R-module. Set

$$I \ := \ \{r \in R \mid \forall m \in M,\, rm = 0\}$$

so that I is the annihilator of M. Prove that R/I is a noetherian ring.

(8) [10] Is it possible to have a field extension $F \subset K$ with degree 2, $[K\colon F] = 2$, where both fields are isomorphic to the field $\mathbb{Q}(x)$ of rational functions in one variable? Either exhibit such an extension or prove that it is impossible.

(9) [18] Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$.

(a) Let E be the splitting field of f over $\mathbb{Q}$. Show that E contains both $\mathbb{Q}(\sqrt[5]{2})$ and $\mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/5}$. What is $[E:\mathbb{Q}]$?

(b) Prove that there exist $\sigma, \tau \in \mathrm{Gal}(E/\mathbb{Q})$ such that

(i) $\sigma(\sqrt[5]{2}) = \zeta\sqrt[5]{2}$ and $\sigma(\zeta) = \zeta$.

(ii) $\tau(\sqrt[5]{2}) = \sqrt[5]{2}$ and $\tau(\zeta) = \zeta^2$.

Use this to show that every element in $\mathrm{Gal}(E/\mathbb{Q})$ can be expressed uniquely as $\sigma^i\tau^j$ for $0 \le i \le 4$ and $0 \le j \le 3$. Hint: Note that every automorphism is determined by its action on $\sqrt[5]{2}$ and ζ. Show that these automorphisms act differently on these elements of E.

(c) Let $H \subseteq \mathrm{Gal}(E/\mathbb{Q})$ be the subgroup generated by $\tau\sigma$. What is the fixed field of H?