Algebra Qualifying Examination
January 6, 2020

Instructions:

- Read all problems first; make sure that you understand them, and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit awarded will be based on the correctness of your answers, as well as the clarity and main steps of your reasoning. Answers must be legible and written in a structured and understandable manner. Do scratch work on a separate page.
- Start each problem on a new page, clearly marking the problem number on that page.
- Rings always have an identity $1 \neq 0$, and all modules are left modules.
- Throughout, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers.

1. Let H be a subgroup of a group G. Consider the *normalizer* and *centralizer* (respectively) of H:

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\} \qquad \text{and} \qquad C_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

   (a) *(5 points)* Prove that both the normalizer and centralizer are subgroups of G.
   (b) *(5 points)* Prove that the centralizer is a *normal* subgroup of the normalizer.
   (c) *(5 points)* Prove that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$ (the group of *automorphisms* of H, that is, bijective group homomorphisms from H to itself).
   (d) *(5 points)* Assume additionally that H is a *normal* subgroup of G, and that H is finite. Prove that the *index* of $C_G(H)$ in G is finite.

2. *(10 points)* Recall that by definition, a commutative ring R is *local* if R has a unique maximal ideal. Prove that a commutative ring R is local if and only if for all $r, r' \in R$, if $r + r' = 1_R$ then r or r' is a unit.

3. *(10 points)* Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Let $\mathrm{id}_A, \mathrm{id}_C$ denote the identity maps on A, C, respectively. Consider the following statements:
   (i) There is an R-module homomorphism $\phi : C \to B$ such that $\beta \circ \phi = \mathrm{id}_C$.
   (ii) There is an R-module homomorphism $\psi : B \to A$ such that $\psi \circ \alpha = \mathrm{id}_A$.

   Prove that (i) implies (ii). *(Note it is also true that (ii) implies (i).)*

4. Let R be a commutative ring and let M be an R-module. Let $T(M)$ be the set of *torsion* elements of M, that is,

$$T(M) = \{m \in M \mid r \cdot m = 0 \text{ for some nonzero } r \in R\}.$$

   (a) *(5 points)* Prove that if R is an integral domain, then $T(M)$ is an R-submodule of M.
   (b) *(5 points)* Give an example of a ring R and an R-module M for which $T(M)$ is not an R-submodule of M.
   (c) *(5 points)* Let M, N be R-modules, and let $f : M \to N$ be an R-module homomorphism. Prove that $f(T(M)) \subseteq T(N)$.

5. *(10 points)* Let R be a commutative ring and let I, J be ideals of R. Prove that there is an R-module isomorphism $(R/I) \otimes_R (R/J) \cong R/(I + J)$.

6. The goal of this problem is to prove that $\mathbb{C}$ is an algebraically closed field. (So, do NOT use this fact in your solution!)
   (a) *(5 points)* Let $K/\mathbb{R}$ be a finite extension. Prove that if $[K : \mathbb{R}]$ is odd, then $K = \mathbb{R}$.
   (b) *(5 points)* Let $L/\mathbb{R}$ be a finite *Galois* extension of $\mathbb{R}$. Prove that $[L : \mathbb{R}]$ is a power of 2. (*Hint*: Sylow's Theorem)
   (c) *(5 points)* Prove that there is *no* extension $K/\mathbb{C}$ with $[K : \mathbb{C}] = 2$.
   (d) *(5 points)* Let $K/\mathbb{C}$ be any finite extension. Show that there is some finite Galois extension $L/\mathbb{R}$ with $\mathbb{R} \subseteq \mathbb{C} \subseteq K \subseteq L$. Show that $L = \mathbb{C}$, and deduce that $K = \mathbb{C}$.

7. Let F be a finite field, let f be a monic irreducible polynomial in $F[x]$, and let $\alpha \in \overline{F}$ be a root of f. Prove the following:
   (a) *(5 points)* $F(\alpha)$ is the splitting field for f over F, and
   (b) *(5 points)* the set of roots of f is $\{\alpha^{|F|^r} \mid r \geq 1\}$.

8. *(10 points)* Is the symmetric group S_4 the internal direct sum of two or more nontrivial subgroups? Prove your answer.