Algebra Qualifying Examination
January 2025

Instructions:

- Read all problems first; make sure you understand them and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit awarded will be based on the correctness of answers and the clarity and main steps of reasoning. Answers must be legible and written in a structured and understandable manner. Do scratch work on a separate page.
- Start each problem on a new page, clearly marking the problem number on that page.
- Rings always have an identity 1 and all modules are left modules.
- Throughout, the integers are denoted $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$.

1. [12 points] Let $\phi : G \to H$ be a group homomorphism. Show that ϕ is injective if, and only if, it satisfies the following property: For every group K and every pair of group homomorphisms $\alpha, \beta : K \to G$, if $\phi \circ \alpha = \phi \circ \beta$ then $\alpha = \beta$.

2. [14] Let G be a group of order 380. Prove that G is not simple. (Note that $380 = 2^2 \cdot 5 \cdot 19$.)

3. [10] Let R be a ring with $1 \neq 0$ for which $x^2 = x$ for all $x \in R$. Prove that R is commutative and is of characteristic 2.

4. [14] Let R be an integral domain. Let S be a multiplicative subset of R (that is, $1 \in S$ and $ss' \in S$ whenever $s, s' \in S$) for which $0 \notin S$.
   (a) Prove that $S^{-1}R$ is isomorphic to a subring of the field of fractions of R.
   (b) Illustrate part (a) with the example $R = \mathbb{Z}$, $S = \{3^i \mid i \geq 0\}$: Briefly describe $S^{-1}R$ as a subring of $\mathbb{Q}$.
   (c) Prove that if R is a principal ideal domain, then every ideal of $S^{-1}R$ is principal.

5. [12] Let A be an abelian group. Let n be an integer, $n \geq 2$.
   (a) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$ is isomorphic to the subgroup of A consisting of all elements a for which $na = 0$.
   (b) Prove that $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} A \;\cong\; A/nA$.

6. [12] Let $V = \mathbb{R}^2$ and $T : V \to V$ be the linear transformation given by orthogonal projection onto the line $y = x$. Consider V to be an $\mathbb{R}[x]$-module via

$$p(x) \cdot v = (p(T))(v)$$

for all $v \in V$ and $p(x) \in \mathbb{R}[x]$.

(a) Find all $\mathbb{R}[x]$-submodules of V.

(b) Is V a cyclic $\mathbb{R}[x]$-module (that is, generated by one element)? Justify your answer.

(c) Is V a direct sum of two proper nonzero $\mathbb{R}[x]$-submodules? Justify your answer.

7. [12] Let F be a field. Let $f(x),\ g(x) \in F[x]$ with $f(x)$ irreducible of degree n. Prove that every irreducible factor of $f(g(x))$ has degree divisible by n.

8. [14] Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$. Let K be the splitting field of $f(x)$.

(a) Find $[K : \mathbb{Q}]$.

(b) Let $G = \mathrm{Gal(K/\mathbb{Q})}$. Let P be a Sylow 2-subgroup of G. Is P normal in G? Justify your answer.