# Applied/Numerical Analysis Qualifying Exam

August 14, 2013

**Cover Sheet – Applied Analysis Part**

**Policy on misprints:** The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name ______________________________________________

**Combined Applied Analysis/Numerical Analysis Qualifier**

**Applied Analysis Part**

**August 14, 2013**

**Instructions:** *Answer any **4** out of the **5** problems on this part of the exam. Show all your work clearly. Please indicate which of the 5 problems you are skipping.*

1. Suppose $X$ and $Y$ are finite dimensional linear spaces with respective bases $\{x_k\}_{k=1}^m$ and $\{y_j\}_{j=1}^n$, and suppose $L : X \to Y$ is a linear operator.

a. Explain how $L$ can be characterized by a matrix $A$.

b. Show that if $Y = X$ (and $\{x_k\}_{k=1}^m$ is used for both) then $\det L := \det A$ is well-defined in the following sense: if $\{x'_k\}_{k=1}^m$ is any other basis for $X$, and $A'$ characterizes $L$ with respect to this basis, then $\det A = \det A'$.

2. Answer the following.

a. State the Weierstrass Approximation Theorem.

b. Show that the collection of all polynomials of the form

$$x(1-x)p(x),$$

where $p$ is a polynomial, is dense in the set

$$\mathcal{S} := \{f \in C^1([0,1]) : f(0) = f(1) = 0\},$$

using the $C^1([0,1])$ norm.

3. Let $\eta > 0$ and consider the operator

$$Lu := \int_{-\infty}^{+\infty} e^{-\eta|x-y|} u(y)dy.$$

Determine whether or not $L$ is bounded as a map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and prove your assertion.

4. Consider the functional

$$E(u) = \int_0^1 u(x)^2 + u'(x)^2 + u''(x)^2 dx,$$

defined on the spaces

$$\mathcal{S} = \{u \in C^4([0,1]) : u(0) = a, u(1) = b, u'(0) = c, u'(1) = d\},$$

and

$$\mathcal{S}_0 = \{u \in C^4([0,1]) : u(0) = 0, u(1) = 0, u'(0) = 0, u'(1) = 0\}.$$

a. Explain the difference between the Fréchet derivative of $E$ at $u \in \mathcal{S}_0$ and the Gâteaux derivative of $E$ at the same point.

b. Compute $E'(u)$ (the variational derivative of $E$) and identify the associated gradient $\frac{\delta E}{\delta u}$. Use your result to derive an ordinary differential equation for the critical points of $E$. (You do not need to solve your equation.)

c. Compute $E''(u)$, and use your result to categorize the critical points from (a) as maxima, minima, or saddle points.

You can assume that $E$ is twice Fréchet differentiable.

5. Consider the operator

$$Lu = (x+1)u'' + xu' - u$$

with domain

$$\mathcal{D}(L) = \{u \in L^2([0,1]) : Lu \in L^2([0,1]), u(0) = 0, u(1) = 0\}.$$

Notice that $u_1(x) = x$ and $u_2(x) = e^{-x}$ both solve $Lu = 0$ (though neither is in $\mathcal{D}(L)$).

a. Compute the adjoint operator $L^*$, along with adjoint boundary conditions. Is $L$ self-adjoint?

b. Compute the Green's function for $L$.

c. Is $L^{-1}$ a compact operator? Explain why or why not.

d. Is $L$ a compact operator? Explain why or why not.

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# NUMERICAL ANALYSIS QUALIFIER

August 14, 2013

**This test is long and you may not have time to finish all problems. It is important that you complete those problem parts for which you can provide mathematically precise solutions. Write concisely and void rambling discussions.**

**Problem 1.** Let $V$ be a closed subspace of $H^1(\Omega)$, $V_h \subset V$ be a finite element approximation space and $\Omega$ a domain in $\mathbb{R}^d$. Given $W^0 \in V_h$, we consider forward Euler approximation: $W^{n+1} \in V_h$, $n = 0, 1, \dots$ satisfying

$$((W^{n+1} - W^n)/k, \theta) + A(W^n, \theta) = (f^n, \theta), \qquad \text{for all } \ \theta \in V_h.$$

Here $k > 0$ is the time step size, $t_n = nk$, $f^n \in V_h$, $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$, $\|\cdot\|$ is the corresponding norm and $A(\cdot, \cdot)$ is a symmetric, coercive, and bounded bilinear form on $V$.

Let $\{\psi_i\}$, $i = 1, \dots, M$ be an orthonormal basis with respect to $(\cdot, \cdot)$ for $V_h$ of eigenfunctions satisfying

$$A(\psi_i, \theta) = \lambda_i(\psi_i, \theta), \qquad \text{for all } \ \theta \in V_h.$$

(a) Expand

$$W^n = \sum_{i=1}^{M} c_i^n \psi_i \qquad \text{and} \qquad f^n = \sum_{i=1}^{M} d_i^n \psi_i$$

and set $\delta_i = 1 - k\lambda_i$. Derive a recurrence relation for $c_i^{n+1}$ in terms of $\delta_i$, $c_i^n$, $k$ and $d_i^n$.

(b) Assume that the CFL condition, $k\lambda_i \le 2$ , holds for all eigenvalues $\lambda_i$. Show that

$$|c_i^n| \le \begin{cases} |c_i^0| & \text{if} \quad f^n = 0, \text{ for all } n, \\ t_n^{1/2} \Big( k \displaystyle\sum_{j=0}^{n-1} |d_i^j|^2 \Big)^{1/2} & \text{if} \quad W^0 = 0, \end{cases}.$$

(c) Use Part (b) above and superposition principle to derive the stability estimate

$$\|W^n\| \le \|W^0\| + t_n^{1/2} \Big( k \sum_{j=0}^{n-1} \|f^j\|^2 \Big)^{1/2}.$$

**Problem 2.** In this problem, $C$ (with or without subscript) denotes generic positive constants which are independent of the triangle diameters $h_\tau$ and $\mathbb{P}^j$ denotes the space of polynomials on $\mathbb{R}^2$ of degree at most $j$.

Let $\Omega$ be a polygonal domain in $\mathbb{R}^2$ and $u$ be the solution in $H_0^1(\Omega)$ of

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx, \qquad \text{for all } \ v \in H_0^1(\Omega), \tag{2.1}$$

with $f \in L^2(\Omega)$.

Let $\mathcal{T}_h$ for $0 < h < 1$ be shape regular trangulations of $\Omega$. Set

$$X_h = \{v_h \in H^1(\Omega) \ : \ v_h|_\tau \in \mathbb{P}^2\} \qquad \text{and} \qquad V_h = X_h \cap H_0^1(\Omega).$$

(a) For $\tau \in \mathcal{T}_h$, let $b_\tau \in \mathbb{P}^3$ be the "bubble" function defined by the conditions $b_\tau$ equals one on the barycenter of $\tau$ and $b_\tau$ vanishes on $\partial\tau$. Show that for any function $w_h \in X_h$

$$C_2 \|w_h\|_{L^2(\tau)} \ge \|b_\tau^{1/2} w_h\|_{L^2(\tau)} \ge C_1 \|w_h\|_{L^2(\tau)}.$$

(b) For $f_h \in X_h$ and $v_h \in V_h$, let $z_h$ denote the function given by

$$z_h|_\tau = b_\tau (f_h + \Delta v_h)|_\tau, \qquad \tau \in \mathcal{T}_h.$$

Explain why $z_h \in H_0^1(\Omega)$.

(c) Show that

$$\|b_\tau^{1/2}(f_h + \Delta v_h)\|_{L^2(\tau)}^2 = \int_\tau (f_h - f) z_h \, dx + \int_\tau (\nabla(u - v_h)) \cdot \nabla z_h \, dx.$$

(d) Show that

$$\|f_h + \Delta v_h\|_{L^2(\tau)} \le C\big(h_\tau^{-1} \|\nabla(u - v_h)\|_{L^2(\tau)} + \|f - f_h\|_{L^2(\tau)}\Big).$$

**Problem 3.** Let $K$ be a nondegenerate triangle in $\mathbb{R}^2$. Let $a_1$, $a_2$, $a_3$ be the three vertices of $K$. Let $a_{ij} = a_{ji}$ denote the midpoint of the segment $(a_i, a_j)$, $i, j \in \{1, 2, 3\}$. Let $\mathbb{P}^2$ be the set of the polynomial functions over $K$ of total degree at most 2. Let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_{12}, \sigma_{23}, \sigma_{31}\}$ be the functionals (or degrees of freedom) on $\mathbb{P}^2$ defined as

$$\sigma_i(p) = p(a_i),\ i \in \{1, 2, 3\} \qquad \sigma_{ij}(p) = p(a_i) + p(a_j) - 2p(a_{ij}),\ i, j = 1, 2, 3,\ i \neq j.$$

(a) Show that $\Sigma$ is a unisolvent set for $\mathbb{P}^2$ (this means that any $p \in \mathbb{P}^2$ is uniquely determined by the values of the above degrees of freedom applied to $p$).
(b) Compute the "nodal" basis of $\mathbb{P}^2$ which corresponds to $\{\sigma_1, \dots, \sigma_{31}\}$.
(c) Evaluate the entry $m_{11}$ of the element mass matrix.

Hint. If you have problem computing the functions from (b) for a general triangle, derive them for a reference element.