Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 8, 2017

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let $L[u] = -\frac{d^2u}{dx^2}$, $0 \le x \le 1$. Take

$$\mathcal{D}(L) := \{u \in L^2[0,1] \,|\, u'' \in L^2[0,1],\ u(0) = 0,\ u'(1) = 3u(1)\}.$$

to be the domain of L.

(a) Show that L is self adjoint on $\mathcal{D}(L)$.
(b) Find the Green's function for the problem $L[u] = f$, $u \in \mathcal{D}(L)$.
(c) Let $Kf(x) := \int_0^1 G(x,y)f(y)dy$. Show that K is a self-adjoint Hilbert-Schmidt operator, and that 0 is not an eigenvalue of K.
(d) Use (b) and the spectral theory of compact operators to show the orthonormal set of eigenfunctions for L form a complete set in $L^2[0,1]$.

Problem 2. Let f be a piecewise smooth, continuous 2π periodic function having a piecewise continuous derivative, f'. Suppose that f has the Fourier series $f(x) = \sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx)$.

(a) Show that it is permissible to interchange sum and derivative to obtain the the Fourier series for f'; that is,

$$f'(x) = \frac{d}{dx}\left\{\sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx))\right\} = \sum_{n=1}^{\infty} n(a_n \cos(nx) - b_n \sin(nx)).$$

(b) Use this result to calculate the Fourier series for the 2π-periodic extension of $f(x) = \frac{\pi x^2 - \pi^2 x}{8}$, $0 \le x \le \pi$, given that $\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1} = \frac{\pi}{4}\,\mathrm{sign}(x)$ on $0 < |x| < \pi$.
(c) Find the $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$

Problem 3. Do the following.

(a) State the Projection Theorem.
(b) State and prove the Fredholm Alternative (Hilbert space version).
(c) Let $k(x,y) = x^3 y$, $Ku(x) = \int_0^1 k(x,y)u(y)dy$, and $Lu = u - \lambda Ku$.
  (i) Briefly explain why L has closed range.
  (ii) Determine the values of λ for which $Lu = f$ has a solution for all f.
  (iii) Solve $Lu = f$ for these values of λ.

Problem 4. Let $I(\lambda) := \int_0^{\infty} e^{-\lambda t} f(t)dt$. Prove this version of Watson's lemma: *Suppose that there are positive constants C_0 and C_1 such that for $0 \le t \le 1$, $|f(t) - a_0 - a_1 t| \le C_0 t^2$, and that for $t \ge 1$, $|f(t)| \le C_1$. Then, $I(\lambda) = a_0\lambda^{-1} + a_1\lambda^{-2} + \mathcal{O}\{\lambda^{-3}\}$.*

APPLIED MATHEMATICS/NUMERICAL ANALYSIS QUALIFIER

August 8, 2017

Numerical Analysis part, 2 hours

Problem 1. Let K be a non-degenerate triangle in $\mathbb{R}^2$. Let a_1, a_2, a_3 be the three vertices of K. Let $a_{ij} = a_{ji}$ denote the midpoint of the segment (a_i, a_j), $i, j \in \{1, 2, 3\}$. Let $\mathbb{P}^1$ be the set of linear functions $p(x_1, x_2)$ over K and $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ be the linear forms (or degrees of freedom) on $\mathbb{P}^1$ defined as

$$\sigma_{ij}(p) = p(a_{ij}),\ i, j = 1, 2, 3,\ i \neq j.$$

(a) Show that the degrees of freedom $\{\sigma_{12}, \sigma_{23}, \sigma_{31}\}$ are unisolvent.
(b) Compute the "nodal" basis of $\mathbb{P}^1$ which corresponds to $\Sigma = \{\sigma_{12}, \sigma_{23}, \sigma_{31}\}$.
(c) Let $\mathcal{T}_h$ be a triangulation of the domain Ω with polygonal boundary and let the finite dimensional space $\mathbb{V}$ consist of functions whose restrictions to each K are the functions from the FE $(K, \mathbb{P}^1, \Sigma)$. Show that in general these functions are NOT in $H^1(\Omega)$.
(d) If M_K is the element "mass" matrix, evaluate its entries m_{ij}.

Problem 2. (a) Let $\Omega = (0, 1)$. Assume that $u \in H^1(\Omega)$ and let $x_0 \in \overline{\Omega}$. Prove that

$$\|u\|^2_{L_2(\Omega)} \leq C_1\Big(u^2(x_0) + \|u'\|^2_{L_2(\Omega)}\Big) \tag{2.1}$$

with a constant C_1 independent of x_0.
(b) Consider the fourth-order boundary value problem

$$u'''' = f \text{ in } \Omega,\ u(0) = 0,\ u''(0) = 0,\ u''(1) + u'(1) = 1,\ u'''(1) = 0.$$

Derive a weak formulation of this problem assuming that $f \in L_2(\Omega)$.
(c) Show that the weak formulation that you derived in part (b) above has a unique solution.
(d) Using Hermite cubic finite element spaces (i.e., piecewise cubic elements lying in $C^1(\Omega)$) derive a finite element method for the problem in part (b). Be sure to carefully define your finite element space.
(e) Show that the finite element method you derived has a unique solution u_h and derive an optimal-order error estimate for $u - u_h$ in the $H^2(\Omega)$-norm. *Hint:* A correct proof will involve using an interpolation error bound. You may state and use such a bound without proving it.

Problem 3. Let $u(x, t)$ be a smooth solution satisfying

$$\partial_t u + \beta \partial_x u = 0, \quad x \in \Omega := (0, 1), \quad t > 0 \quad \text{and} \quad u(0, x) = \phi(x), \quad x \in \Omega$$

where $\beta \in \mathbb{R}$ and ϕ is a given smooth function. In addition, we assume that $u(x, t)$ satisfies the periodic boundary condition $u(0, t) = u(1, t)$, $t > 0$. Let $\mathbb{V} = \{v \in H^1(\Omega) :\ v(0) = v(1)\}$.

(a) Let $N \in \mathbb{N}\backslash\{0\}$, set $h := \frac{1}{N+1}$ and consider the uniform mesh $\mathcal{T}_h$ composed of the cells $[x_i, x_{i+1}]$, $i = 0, ..., N$. Let $P(\mathcal{T}_h)$ be the finite element space composed of continuous piecewise linear functions on $\mathcal{T}_h$. Given $\phi_h \in \mathbb{V} \cap \mathcal{P}(\mathcal{T}_h)$ an approximation of ϕ, consider the semi-discrete method: For $t > 0$, find $u_h(t, .) \in \mathbb{V} \cap \mathcal{P}(\mathcal{T}_h)$ such that $u_h(0, x) = \phi_h(x)$ and for every $v_h \in \mathcal{P}(\mathcal{T}_h)$ with $v_h(0) = v_h(1)$ there holds

$$\frac{h}{2}\sum_{i=0}^{N} \left(\partial_t u_h(t, x_{i+1}) v_h(x_{i+1}) + \partial_t u_h(t, x_i) v_h(x_i)\right) + \beta \int_\Omega \partial_x u_h(t, x) v_h(x)\ dx = 0.$$

Show that the above problem can be reformulated as a system of ODEs and express this system in matrix-vector form.
Note: we assume that as a function of t, $u_h(t) \to \mathbb{V} \cap \mathcal{P}(\mathcal{T}_h)$ is smooth.

(b) Show that the Finite Element approximation $u_h(t)$ satisfies

$$\frac{d}{dt}\sum_{i=0}^{N} u_h(t, x_i)^2 = 0.$$

(c) Show that

$$c^{-1}\int_{\Omega} u_h^2(t,x)\ dx \leq h\sum_{i=0}^{N} u_h(t,x_i)^2 \leq c\int_{\Omega} u_h^2(t,x)\ dx$$

and deduce the estimate

$$\int_{\Omega} u_h^2(t,x)\ dx \leq C\int_{\Omega} \phi_h^2(0,x)\ dx.$$

Here c and C are constants independent of h.