APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

August 6, 2019

Applied Analysis Part, 2 hours

Name: ______________________________

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let $f \in C[0,1]$, $\delta > 0$, and $\omega(f,\delta)$ be the modulus of continuity for f.

(a) Let $\Delta = \{x_0 = 0 < x_1 < \cdots < x_n = 1\}$ be a knot sequence with norm $\|\Delta\| = \max |x_j - x_{j+1}|$, $j = 0, \ldots, n-1$. If s_f is the linear spline that interpolates f at the x_j's, show that $\|f - s_f\|_\infty \le \omega(f, \|\Delta\|)$.

(b) Using part (a) and the fact that the continuous functions are dense in $L^1[0,1]$, prove the Riemann-Lebesgue Lemma: $\lim_{|\lambda|\to\infty} \int_0^1 g(x)e^{i\lambda x}dx = 0$, for all $g \in L^1[0,1]$.

Problem 2. Let $\mathcal{D}$ be the set of compactly supported C^∞ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.

(b) Consider a function $f \in C^{(1)}(\mathbb{R})$ such that both f and f' are in $L^1(\mathbb{R})$, and $\int_{\mathbb{R}} f(x)dx = 1$. Define the sequence of functions $\{T_n(x) := n^2 f'(nx) \colon n = 1, 2, \ldots\}$. Show that, in the sense of distributions — i.e., in $\mathcal{D}'$ —, T_n converges to δ'.

Problem 3. Let L be a closed, densely defined (possibly unbounded) linear operator on a Hilbert space $\mathcal{H}$, and let the range of L be dense in $\mathcal{H}$.

(a) Show that if there exists $C > 0$ such that $\|Lf\| \ge C\|f\|$ for all $f \in \mathcal{D}$, then L^{-1} is bounded.

(b) Use (a) to show that if $L = L^*$, then the spectrum of L is contained in $\mathbb{R}$.

Problem 4. Consider the boundary problem below::

$$L[u] = \frac{d}{dx}\left(x\frac{du}{dx}\right) = f, \text{ where } \mathcal{D} = \{u \in L^2[1,e] \colon Lu \in L^2[1,e],\ u'(1) = 0,\ u(e) = 0\},$$

(a) Find the Green's function $g(x,y)$ for the problem, given that $1, \log(x)$ solve $L[u] = 0$.

(b) Show that $Kf(x) = \int_1^e g(x,y)f(y)dy$ is self adjoint, and briefly explain why it's compact. Show *directly* from the spectral theory for compact operators that the orthonormal set of eigenfunctions for L is complete in $L^2[1,e]$. (Do not solve the eigenvalue problem.)

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<u>Problem 1.</u> Consider the boundary value problem: Find u such that

$$-\Delta u = f \text{ in } \Omega,\ \nabla u \cdot \mathbf{n} + u = 0 \text{ on } \Gamma, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $\Gamma = \partial\Omega$ is the boundary of Ω, $\mathbf{n}$ is the outward-pointing unit normal on Γ, and $q \in \mathbb{R}$ and $f \in L_2(\Omega)$ are given.
(a) The problem (1) has weak form given by: Find $u \in \mathbb{V}$ such that

$$a(u, v) = L(v),\ \forall v \in \mathbb{V}. \tag{2}$$

Identify the bilinear form a, the linear form L, and the function space $\mathbb{V}$.
(b) Show that the problem (2) has a unique solution.
<u>Hint:</u> If you have correctly identified $\mathbb{V}$, then there holds

$$\|u\|_{L_2(\Omega)} \leq C(\|\nabla u\|_{L_2(\Omega)} + \|u\|_{L_2(\Gamma)}),\ u \in \mathbb{V}.$$

You may use this inequality without proof.
(c) Let $\mathcal{T}_h$ be a shape-regular partition of Ω into triangles. Introduce the finite dimensional space $\mathbb{V}_h$ consisting of continuous piecewise linear polynomials over $\mathcal{T}_h$. Consider the finite element approximation of (2): find

$$u_h \in \mathbb{V}_h, \quad \text{s.t.} \quad a(u_h, v) = L(v) \quad \text{for all} \quad v \in \mathbb{V}_h. \tag{3}$$

State and prove the optimal estimate for the error $\|u - u_h\|_{\mathbb{V}}$ assuming that the solution to (2) belongs to the Sobolev space $H^2(\Omega)$. As part of your proof you should define an appropriate interpolation operator and state, but not prove, optimal error estimates for this operator.
(d) Derive an optimal error bound for $\|u - u_h\|_{L^2(\Omega)}$ under the assumption of full regularity of the problem (2).

<u>Problem 2.</u> Consider the interval $I(0,1)$ and the set of continuous functions $\hat{v}$ defined on $[0,1]$. Let $\hat{a}_1 = 0$, $\hat{a}_2 = 1/4$, and $\hat{a}_3 = 1$. Consider also the following set of degrees of freedom:

$$\Sigma = \{\hat{v}(\hat{a}_1),\ \hat{v}(\hat{a}_3),\ \hat{v}'(\hat{a}_2)\}.$$

(a) Show that triple $(I, \mathbb{P}_2, \Sigma)$ is a finite element.
(b) Write down the basis for the quadratic polynomials $\mathbb{P}_2$ that is dual to Σ, that is, find $q_i \in \mathbb{P}_2$ $(i = 1, 2, 3)$ such that $\hat{q}_i(\hat{a}_j) = \delta_{ij}$ $(i = 1, 2, 3$ and $j = 1, 3)$ and $\hat{q}_i'(\hat{a}_2) = \delta_{i2}$ $(i = 1, 2, 3)$. Then write down the finite element interpolant $\hat{\Pi}(\hat{w})$ of a given function $\hat{w} \in C^0[0,1]$ with respect to the given degrees of freedom.
(c) Consider the interval $[a, b]$, let F map $[0,1]$ onto $[a, b]$, and let $v \in H^3(a, b)$. Define $\Pi(v)$ by $(\Pi(v)) \circ F = \hat{\Pi}(v \circ F)$. Use the Bramble-Hilbert Lemma and the reference map F in order to estimate the error

$$\|v' - \Pi(v)'\|_{L_2(a,b)}$$

in terms of $h = b - a$. Explain how to modify the proof when v is less regular, in particular when $v \in H^2(a, b)$.

Problem 3. Let Ω be a bounded domain and $T > 0$ be a given final time. For $f \in C^0([0, T]; L_2(\Omega))$ and $u_0 \in H_0^1(\Omega)$ given, we consider the parabolic problem consisting in finding $u : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) - \Delta u(x,t) = f(x,t) & \text{for } (x,t) \in \Omega \times (0, T], \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times [0, T], \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

We assume that the solution u to the above problem is sufficiently smooth.

Let N be a strictly positive integer and let $\tau := T/N$, $t_n := n\tau$ and $t^{n+\frac{1}{2}} := \frac{1}{2}(t^{n+1} + t^n)$ for $n = 0, ..., N$. We consider the following semi-discretization in time: Set $U^0 := u_0$ and define $U^n : \Omega \to \mathbb{R}$ recursively by

$$\begin{cases} \frac{1}{\tau}(U^{n+1}(x) - U^n(x)) - \frac{1}{2}\Delta(U^{n+1}(x) + U^n(x)) = f(x, t^{n+\frac{1}{2}}) & \text{for } x \in \Omega, \\ U^{n+1}(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

(1) (Stability) Show that for $n = 0, ..., N$, U^n satisfies

$$\|U^{n+1}\|^2_{L_2(\Omega)} \le \|U^0\|^2_{L_2(\Omega)} + \frac{1}{2} C_p^2 \tau \sum_{j=0}^{n} \|f(t^{j+\frac{1}{2}})\|^2_{L_2(\Omega)}.$$

(2) (Consistency I) Show either (but not both) that

$$\|\frac{1}{\tau}(u(t^{n+1}) - u(t^n)) - \frac{\partial}{\partial t} u(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \le C\tau^{\frac{3}{2}} \|\frac{\partial^3}{\partial t^3} u\|_{L_2(t^n, t^{n+1}; L_2(\Omega))}$$

or

$$\|\frac{1}{2}\Delta\left(u(t^{n+1}) + u(t^n)\right) - \Delta u(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \le C\tau^{\frac{3}{2}} \|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_2(t^n, t^{n+1}; L_2(\Omega))}.$$

Here C is a constant independent of τ, T and u.

Hint: You can use without proof the following Taylor expansion formula

$$g(b) = g(a) + g'(a)(b-a) + ... + \frac{1}{n!} g^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b (b-t)^n g^{(n+1)}(t)dt.$$

(3) (Consistency II) Deduce from the previous item that for a constant C independent of τ, T and u we have

$$\|\frac{1}{\tau}(u^{n+1}(x) - u^n(x)) - \frac{1}{2}\Delta(u^{n+1}(x) + u^n(x)) - f(t^{n+\frac{1}{2}})\|_{L_2(\Omega)}$$
$$\le C\tau^{\frac{3}{2}} \left(\|\frac{\partial^3}{\partial t^3} u\|_{L_2(t^n, t^{n+1}; L_2(\Omega))} + \|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_2(t^n, t^{n+1}; L_2(\Omega))} \right).$$

(4) From (2) and (4), conclude the following estimate for the error $e^n := u(t^n) - U^n$:

$$\|e^N\|^2_{L_2(\Omega)} \le C\tau^4 \left(\|\frac{\partial^3}{\partial t^3} u\|^2_{L_2(0,T; L_2(\Omega))} + \|\frac{\partial^2}{\partial t^2}\Delta u\|^2_{L_2(0,T; L_2(\Omega))} \right),$$

where C is a constant independent of τ, T and u.