Applied/Numerical Analysis Qualifying Exam

January 11, 2011

Name ....

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Part 1: Applied Analysis

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

(1) Given $w \in C[0,1]$, with $w(x) > 0$ on $[0,1]$, let $L^2_w[0,1]$ be the weighted Hilbert space with the inner product

$$\langle f, g\rangle_w = \int_0^1 f(x)\overline{g(x)}w(x)dx,$$

where f, g are in $L^2[0,1]$. In addition, let $\{\phi_n(x)\}_{n=0}^\infty$ be the set of orthogonal polynomials generated by using the Gram-Schmidt process on $\{1, x, x^2 \ldots\}$ in the inner product for L^2_w. Assume that $\phi_n(x) = x^n + \text{lower powers}$.

(a) State the Weierstrass Approximation Theorem and briefly sketch its proof. (Use no more than a page or so.)

(b) You are given that $C[0,1]$ is dense in $L^2[0,1]$. Show that the orthogonal polynomials $\{\phi_n(x)\}_{n=0}^\infty$ form a complete, orthogonal set in $L^2_w[0,1]$.

(2) Consider the differential operator $Lu(x) = -((x+1)u')'$, with $x \in [0,1]$.

(a) Show that if $D(L) := \{u \in L^2 \,|\, Lu \in L^2 \text{ and } u(0) = 0 = u'(1)\}$, then L is self adjoint and positive definite.

(b) Find the Green's function for L having the domain $D(L)$ above.

(c) Briefly explain why the eigenfunctions this operator are complete in $L^2[0,1]$.

(3) In the problem below, use the Fourier transform conventions

$$\mathcal{F}[f](\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$
$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega.$$

As usual, $\hat{f} = \mathcal{F}[f]$.

(a) Show $\mathcal{F}^4 = I$. (Hint: $\mathcal{F}[f(x)] = \mathcal{F}^{-1}[f(-x)]$.)

(b) You are given that the equation $-u_n'' + x^2u_n = (2n+1)u_n$ has, up to a constant multiple, a unique solution $u_n \in L^2(\mathbb{R})$, for $n = 0, 1, \ldots$. (You may assume that the solution is smooth enough and decays fast enough to be in Schwartz space.) Show that u_n is an eigenfunction of the Fourier transform; that is, $\hat{u}_n(\omega) = \lambda_n u_n(\omega)$. Also, show that $\lambda_n^4 = 1$.

(4) Let $k(x,y) = x^4y^{12}$ and consider the operator $Ku(x) = \int_0^1 k(x,y)u(y)dy$.

(a) Show that K is a Hilbert-Schmidt operator and that $\|K\|_{\text{op}} \le \frac{1}{10}$.

(b) State the Fredholm Alternative for the operator $L = I - \lambda K$. Explain why it applies in this case. Find all values of λ such that $Lu = f$ has a unique solution for all $f \in L^2[0,1]$.

(c) Use a Neumann series to find the resolvent $(I - \lambda K)^{-1}$ for λ small. Sum the series to find the resolvent.

Part 2: Numerical Analysis

Instructions: *Do all problems in this part of the exam. Show all of your work clearly.*

Problem 1: Consider the following two-points boundary value second order problem in 1-D: Find a function u defined a.e. in $]0,1[$ such that

$$(1)\qquad \begin{aligned} -\big(xK(x)u'(x)\big)' + xq(x)u(x) &= xf(x) \text{ a.e. in }]0,1[\,,\\ \lim_{x\to 0}\big(xu'(x)\big) = 0 \text{ and } K(1)&u'(1) + u(1) = 0\,, \end{aligned}$$

where $K \in \mathcal{C}^1([0,1])$, $q \in \mathcal{C}^0([0,1])$ and $f \in L^2(0,1)$ are given functions. Assume that there exists a constant $\kappa_0 > 0$ such that $K(x) \geq \kappa_0$ and $q(x) \geq 0$ for all $x \in [0,1]$. Let

$$V = \{v \in L^2_{\text{loc}}(0,1)\,;\ \sqrt{x}v \in L^2(0,1), \sqrt{x}v' \in L^2(0,1)\}\,.$$

Accept as a fact that V is a Hilbert space for the norm

$$\|v\|_V = \Big(\|\sqrt{x}v\|^2_{L^2(0,1)} + \|\sqrt{x}v'\|^2_{L^2(0,1)}\Big)^{1/2},$$

and $\mathcal{C}^1([0,1])$ is dense in V for this norm.

(1) Derive the variational formulation (also called weak formulation) of problem (1) in the space V.

(2) Prove that the corresponding bilinear form of this variational formulation is elliptic (or coercive) in V.
Hint. First show that all functions v of $\mathcal{C}^1([0,1])$ satisfy

$$\int_0^1 v(x)^2 dx = v^2(1) - 2\int_0^1 xv(x)v'(x)dx$$

and then establish the following variant of Poincaré's inequality

$$\forall v \in V\,,\ \|\sqrt{x}v\|_{L^2(0,1)} \leq \alpha\Big(v^2(1) + \|\sqrt{x}v'\|^2_{L^2(0,1)}\Big)^{\frac{1}{2}}$$

for some constant $\alpha > 0$. Based on this equality deduct the ellipticity.

(3) Choose an integer $N \geq 2$, set $h = 1/N$, let $x_i = ih$, $0 \leq i \leq N$ and define the finite element space

$$V_h = \{v_h \in \mathcal{C}^0([0,1])\,;\ v_h|_{]x_i,x_{i+1}[} \in \mathcal{P}_1, 0 \leq i \leq N-1\}\,.$$

Show that V_h is a subspace of V. Discretize the variational problem in this space. Prove existence and uniqueness of the discrete solution and establish an error estimate without estimating the norms of the interpolation errors.

Problem 2: Let Ω be a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$. Let

$$H^1_0(\Omega) = \{v \in H^1(\Omega)\,:\, v(x) = 0\,\forall x \in \partial\Omega\}$$

be the standard Sobolev space of functions defined on Ω that vanish on the boundary.

In all that follows, $T > 0$ is a given final time, $c > 0$ is a constant, and $u_0 \in \mathcal{C}^0(\Omega)$ are given functions. Consider the parabolic equation: Find a function u defined a.e. in $\Omega\times]0,T[$ solution of

$$(2)\qquad \begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + cu &= 0 \text{ a.e. in } \Omega\times]0,T[\,,\\ u(x,t) &= 0 \text{ a.e. in } \partial\Omega\times]0,T[\,,\\ u(x,0) &= u_0(x) \text{ a.e. in } \Omega\,. \end{aligned}$$

Accept as a fact that problem (2) has one and only one solution u in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

Let $\mathcal{T}_h$ be a finite element partition of Ω into triangles τ of diameter $h_\tau \leq h$. Further, let

$$W_h = \{v_h \in \mathcal{C}^0(\bar{\Omega})\,;\, \forall \tau \in \mathcal{T}_h, v_h|_\tau \in \mathcal{P}_1, v_h|_{\partial\Omega} = 0\}\,,$$

be a finite element space of continuous piece-wise linear functions over $\mathcal{T}_h$.

Consider the fully discrete backward Euler implicit approximation of (2): for K a positive integer, set $k = T/K$, define $t_n = nk$, $0 \leq n \leq K$, and for each $0 \leq n \leq K-1$, knowing $u_h^n \in W_h$ find $u_h^{n+1} \in W_h$ such that

$$\text{(3)} \qquad \forall v_h \in W_h\,,\ \frac{1}{k}\big(u_h^{n+1} - u_h^n, v_h\big) + a(u_h^{n+1}, v_h) = 0,\ n = 0, 1, \cdots, K,\ u_h^0 = I_h(u_0).$$

Here $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, the bilinear form $a(u_h^{n+1}, v_h)$ comes from the variational formulation of problem (2), and I_h is the Lagrange interpolation operator in W_h. Write the expression of $a(u_h^{n+1}, v_h)$.

(1) Show that (3) defines a unique function u_h^{n+1} in W_h.

(2) Prove the following stability estimate

$$\text{(4)} \qquad \sup_{1\leq n\leq K} \|u_h^n\|^2_{L^2(\Omega)} + k\sum_{n=1}^{K} |u_h^n|^2_{H^1(\Omega)} \leq \|u_h^0\|^2_{L^2(\Omega)}.$$

(3) Also prove the estimate

$$\text{(5)} \qquad \sup_{1\leq n\leq K} |u_h^n|_{H^1(\Omega)} \leq |u_h^0|_{H^1(\Omega)}.$$

Problem 3: Consider the interval $(0,1)$ and the set of continuous functions $\hat{v}$ defined on $[0,1]$. Let $\hat{a}_1 = 0$, $\hat{a}_2 = \frac{1}{2}$, $\hat{a}_3 = 1$.

(1) Consider the following two sets of degrees of freedom

$$\Sigma_1 = \{\hat{v}(\hat{a}_j),\, j = 1,2,3\} \quad \Sigma_2 = \{\hat{v}(\hat{a}_1),\, \hat{v}(\hat{a}_3),\, \int_0^1 \hat{v}(s)ds\}.$$

Write down the basis functions of $\mathcal{P}_2$ (for both sets of degrees of feedom) such that
(a) $p_i \in \mathcal{P}_2$, $1 \leq i \leq 3$, satisfying: $p_i(\hat{a}_j) = \delta_{i,j}, 1 \leq i,j \leq 3$ for the set Σ_1;
(b) $q_i \in \mathcal{P}_2$, $1 \leq i \leq 3$, satisfying:

$$q_i(\hat{a}_j) = \delta_{i,j},\ \int_0^1 q_i(s)ds = 0, i = 1,3, j = 1,3,$$
$$\int_0^1 q_2(s)ds = 1, q_2(\hat{a}_1) = q_2(\hat{a}_3) = 0, \quad \text{for the set } \Sigma_2.$$

In both cases, write down the FE interpolant $\hat{\Pi}(\hat{w})$ of a given function $\hat{w} \in \mathcal{C}^0([0,1])$.

(2) Consider the interval $[a,b]$, let F map $[0,1]$ onto $[a,b]$, and let v be given in $H^3(a,b)$. Define $\Pi(v)$ by $\big(\Pi(v)\big) \circ F = \hat{\Pi}\big(v \circ F\big)$. Give the Bramble Hilbert argument to get an estimate in terms of $h = b - a$ for the error

$$\|v' - \Pi(v)'\|_{L^2(a,b)}\,.$$

Explain how to modify the proof when v is less regular, e.g $v \in H^2(a,b)$.