APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

January 10, 2019

Applied Analysis Part, 2 hours

Name: ____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: *Do any four problems. Show all work clearly. State the problem that you are skipping.* No *extra credit for doing all five.*

Problem 1. Let A be an $n \times n$ self-adjoint matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) State the Courant-Fischer mini-max theorem.

(b) Let $B = [b_1\ b_2]$ be a real $n \times 2$ matrix, with b_1, b_2 being linearly independent. Assume that $\|x\| = 1$. If $q(x) = x^T Ax$ and $\hat{q}(x) = q(x)|_{B^T x=0}$, show that

$$\lambda_3 \leq \max_{\|x\|=1} \hat{q}(x) \leq \lambda_1.$$

Problem 2. A sequence $\{f_n\}$ in $\mathcal{H}$ is said to be weakly convergent to $f \in \mathcal{H}$ if and only if $\lim_{n\to\infty}\langle f_n, g\rangle = \langle f, g\rangle$ for every $g \in \mathcal{H}$. When this happens, we write $f = w\text{-}\lim_{n\to\infty} f_n$. One can show that every weakly convergent sequence is bounded.

(a) Let $\{\phi_n\}_{n=1}^{\infty}$ be any orthonormal sequence. Show that $w\text{-}\lim_{n\to\infty} \phi_n = 0$. (Hint: use Bessel's inequality.)

(b) Let K be a compact linear operator on a Hilbert space $\mathcal{H}$. Show that if $w\text{-}\lim_{n\to\infty} f_n = f$, then $\lim_{n\to\infty} Kf_n = Kf$.

(c) Define $\rho(K)$, the resolvent set for K, and $\sigma(K)$, the spectrum of K. Use (a) and (b) to show that $0 \in \sigma(K)$.

Problem 3. Let $J[y] := \int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right)dx$. Find the extremal of J that satisfies natural boundary conditions at $x = 0$ and $x = 1$.

Problem 4. Consider the operator $Lu = x^2u'' - xu'$ with domain $\mathcal{D}_L := \{u \in L^2[1,2] : Lu \in L^2[1,2],\ u(1) = 0\ \&\ u'(2) = 0\}$. You are given that the homogenous solutions of $Lu = 0$ are 1 and x^2, neither of which is in $\mathcal{D}_L$.

(a) Compute the adjoint L^*, along with the adjoint boundary conditions. Is L self adjoint?

(b) Compute the Green's function for L.

(c) Is L^{-1} compact? Justify your answer.

Problem 5. State and sketch a proof of the Weierstrass Approximation Theorem.

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Question I.

Consider the variational problem: find

$$u \in H^1(\Omega) \equiv \mathbb{V}, \quad \text{s.t. } a(u,v) = L(v) \text{ for all } v \in \mathbb{V} \equiv H^1(\Omega). \tag{1}$$

Here $\Omega = (0,1) \times (0,1)$, $\Gamma = \partial\Omega$ is its boundary,

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Gamma uv \, \mathrm{d}s, \quad \text{and} \quad L(v) = \int_\Gamma gv \, \mathrm{d}s, \tag{2}$$

where g is a given smooth function of Γ.

(a) Derive the strong form of problem (1).

(b) Let $\mathcal{T}_h$ be a shape-regular partitioning of Ω into triangles. Introduce the finite dimensional space $\mathbb{V}_h$ consisting of continuous piecewise linear polynomials over $\mathcal{T}_h$. Show that $\mathbb{V}_h \subset \mathbb{V}$.

(c) Consider the finite element approximation of (1): find

$$u_h \in \mathbb{V}_h, \quad \text{s.t.} \quad a(u_h, v) = L(v) \quad \text{for all} \quad v \in \mathbb{V}_h. \tag{3}$$

State (not prove) the optimal estimate for the error $\|u - u_h\|_{\mathbb{V}}$ assuming that the solution to (1) belongs to the Sobolev space $H^2(\Omega)$. Derive a bound for $\|u - u_h\|_{L^2(\Omega)}$ under the assumption of full regularity of the problem (1).

(d) Assume that in the evaluation of the boundary term $\int_\Gamma u_h v ds$ you have applied the composite trapezoidal quadrature rule:

$$\int_\Gamma f \, \mathrm{d}s \approx \sum_{e\in\Sigma} \frac{|e|}{2}(f(e_1) + f(e_2)) := \sum_{e\in\Sigma} Q_e(f),$$

where Σ is the set of boundary edges and for $e \in \Sigma$, e_1, e_2 are the endpoints of e (order is irrelevant) and $|e|$ is the length of e. In this way you have generated the approximate bilinear from

$$a_h(u_h, v) = \int_\Omega \nabla u_h \cdot \nabla v \, \mathrm{d}x + \sum_{e\in\Sigma} Q_e(u_h v).$$

State the FEM using this approximation (this is one of the cases of variational "crimes"). Show that

$$a_h(v_h, v_h) \geq c\|v_h\|^2_{\mathbb{V}}, \qquad \forall v_h \in \mathbb{V}_h,$$

where c is a constant only depending on Ω.
Hint: Recall that there exists a constant C only depending on Ω such that for all $v \in \mathbb{V}$

$$C\int_\Omega v^2 \leq \int_\Omega |\nabla v|^2 + \int_\Gamma v^2.$$

(e) Show that

$$|a(u_h, v) - a_h(u_h, v)| \leq Ch\|u_h\|_{\mathbb{V}}\|v\|_{\mathbb{V}} \quad \text{for} \quad u_h, v \in \mathbb{V}_h,$$

where C is a constant only depending on Ω.

Question II.

Consider the following initial boundary value problem: find $u(\cdot,t) := u(t) \in \mathbb{V}$, with $\mathbb{V} := H_0^1(\Omega)$, s.t.

$$(\frac{d}{dt}u(t), \phi) + (\nabla u(t), \nabla \phi) = (f(t), \phi), \quad \forall \phi \in \mathbb{V}, \quad t > 0, \quad u(x,0) = u_0(x), \quad x \in \Omega, \tag{4}$$

where $u_0 : \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ are given functions and $f(t) := f(\cdot, t)$.

Let $\mathbb{V}_h \subset \mathbb{V} := H_0^1(\Omega)$ consists of continuous piecewise linear functions over a partition $\mathcal{T}_h$ of Ω into triangles.

(a) Consider the semi-discrete (in space) Galerkin finite element approximation of (4): find $u_h(t) \in \mathbb{V}_h$ s.t.

$$(\frac{d}{dt}u_h(t), \phi) + (\nabla u_h(t), \nabla \phi) = (f(t), \phi), \quad \forall \phi \in \mathbb{V}_h, \quad t > 0, \quad u_h(0) = R_h u_0, \tag{5}$$

where $R_h u_0 \in \mathbb{V}_h$ satisfies

$$(\nabla R_h u_0, \nabla \phi) = (\nabla u_0, \nabla \phi), \qquad \forall \phi \in \mathbb{V}_h.$$

Prove that the solution $u_h(t)$ satisfies the a priori estimate

$$\|u_h(t)\|^2 \leq \|u_h(0)\|^2 + c_0 \int_0^t \|f(s)\|^2 ds, \quad t > 0, \tag{6}$$

where c_0 is the constant in the Poincaré inequality $\|v\|^2 \leq c_0 \|\nabla v\|^2$.

(b) Let $k > 0$ and set $t_n = nk$ for $n = 0, 1, ...$ The implicit Euler scheme approximating the problem (5) is given by: Set $U^0 = R_h u(0) = u_h(0)$, find $U^n \in \mathbb{V}_h$ recursively such that for $n = 1, \ldots$ it satisfies

$$\left(\frac{U^n - U^{n-1}}{k}, \phi\right) + (\nabla U^n, \nabla \phi) = (f(t_n), \phi), \ \forall \phi \in \mathbb{V}_h.$$

Prove an a priori estimate for this fully discrete method that is similar to estimate (6):

$$\|U^n\|^2 \leq \|U^0\|^2 + c_0 \sum_{j=1}^{n} k \|f(t_j)\|^2.$$

Derive an a priori estimate for the error $e = u_h(t_n) - U^n$.

Question III.

Let Q be the three dimensional cube

$$Q = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_i \leq 1, \quad i = 1, 2, 3\right\},$$

and let $\mathcal{Q}_2$ be the space of polynomials of degree 2 **in each direction**. Consider the point value evaluation functionals defined for any $p \in \mathcal{Q}_2$

$$\sigma_{i,j,k}(p) = p(i/2, j/2, k/2)$$

for $i, j, k = 0, 1, 2$ Show that this choice of Q, $\mathcal{Q}_2$, and degrees of freedom $\{\sigma_{i,j,k}\}$ is unisolvent.
Hint: you can use without proof the following result:

> *Let p be a polynomial of degree $d \geq 1$ that vanishes on the hyperplane given by the relation $h(x) = 0$. Then $p(x) = h(x)q(x)$, where q is a polynomial of degree $d - 1$.*