Applied Analysis Part

January 8, 2025

Name: ____

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.

(b) Show that $\psi \in \mathcal{D}$ satisfies $\psi(x) = (x\phi(x))'$ for some $\phi \in \mathcal{D}$ if and only if

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \text{ and } \int_{0}^{\infty} \psi(x)dx = 0.$$

(c) Find all distributions $T \in \mathcal{D}'$ such that $xT' = 0$.

Problem 2. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) State and prove the Fredholm Alternative.

(b) Let $\mathcal{H} = L^2[0,1]$. Define the kernel $k(x,y) := x^4y^2$ and let $Ku(x) = \int_0^1 k(x,y)u(y)dy$. Show that K is in $\mathcal{C}(\mathcal{H})$.

(c) Let $L = I - \lambda K$, $\lambda \in \mathbb{C}$, with K as defined in part (b) above. Find all λ for which $Lu = f$ *can* be solved for all $f \in L^2[0,1]$. For these values of λ, find the resolvent $(I - \lambda K)^{-1}$.

Problem 3. Let $L[u] = -u''$, $u(0) = 0,\ u'(1) + u(1) = 0$.

(a) Show that L is self adjoint.

(b) Find the Green's function, $g(x,y)$, for L

(c) Show that $Gu(x) = \int_0^1 g(x,y)u(y)dy$ is a self adjoint compact operator.

(d) Use the spectral theory for compact operators and part (c) to show that from among the eigenfunctions[1] of L we may select a complete orthonormal set for $L^2[0,1]$.

Problem 4. Let f be a continuously differentiable, 2π periodic function having the Fourier series $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$. The trapezoidal rule for numerically finding $\int_0^{2\pi} f(t)dt$ is given by

$$Q_n(f) = \frac{2\pi}{n} \sum_{k=0}^{n-1} f(2\pi k/n).$$

(a) Consider the partial sum $S_{n-1}(t) = \sum_{k=-(n-1)}^{n-1} c_k e^{ikt}$. Show that $Q_n(S_{n-1}) = \int_0^{2\pi} f(t)dt$.

(b) Show that $\left|Q_n(f) - \int_0^{2\pi} f(t)dt\right| \le 2\pi \|f - S_{n-1}\|_{C[0,2\pi]}$.

(c) Suppose that $|c_k| \le |k|^{-6}$ for all $k \neq 0$. Estimate $\left|Q_n(f) - \int_0^{2\pi} f(t)dt\right|$.

[1] Do *not* find the eigenvalues or eigenfunctions.

Name ____

APPLIED MATHEMATICS QUALIFIER: NUMERICAL ANALYSIS PART

Problem 1. For this problem, you may use without proof Poincaré inequality and interpolation estimates as long as you accurately state them.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $f \in L_2(\Omega)$. Consider the function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv, \qquad \forall v \in H_0^1(\Omega).$$

(1) State an additional assumption under which for every $f \in L_2(\Omega)$, we have $u \in H^2(\Omega)$ and $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L_2(\Omega)}$ for a constant C only depending on Ω. From now on we assume that such assumption holds.

(2) Consider a shape-regular and quasi-uniform sequence of triangulation $\{\mathcal{T}_h\}_{h>0}$ of Ω and design a conforming finite element approximation u_h of u such that

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|f\|_{L_2(\Omega)},$$

where C only depends on Ω, the shape-regularity and quasi-uniformity constants. Justify your answer by proving the above estimate.

(3) Now let $z \in H_0^1(\Omega)$ be given by the relations

$$\int_\Omega \nabla z \cdot \nabla v = \int_\Omega v, \qquad \forall v \in H_0^1(\Omega).$$

Using a duality-type argument involving z show that

$$\int_\Omega (u - u_h) \leq Ch^2\|1\|_{L_2(\Omega)}\|f\|_{L_2(\Omega)},$$

where C only depends on Ω, the shape-regularity and quasi-uniformity constants. Be sure to clearly justify all the steps.

Problem 2. Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $T > 0$, $f \in C^0(0,T;L_2(\Omega))$ and $u_0 \in L_2(\Omega)$. Consider the solution u to the parabolic problem

$$\frac{\partial}{\partial t}u - \Delta u = f \qquad \text{in } \Omega \times (0,T], \quad u = u_0 \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \times (0,T].$$

We assume that u is sufficiently smooth. We equip $H_0^1(\Omega)$ with the norm $\|v\|_{H_0^1(\Omega)} := \|\nabla v\|_{L_2(\Omega)}$ and let $H^{-1}(\Omega)$ be its dual space.

For $N \in \mathbb{N}$ and $\frac{1}{2} \leq \theta \leq 1$, consider the θ-method for the time approximation: Let $u^0 = u_0$ and for $n = 1, ..., N$, Define recursively $u^n \in H_0^1(\Omega)$ as the solution to

$$\frac{1}{\tau}\int_\Omega (u^n - u^{n-1})v + \int_\Omega (\theta\nabla u^n + (1-\theta)\nabla u^{n-1})\cdot\nabla v = \int_\Omega (\theta f(t_n) + (1-\theta)f(t_{n-1}))v, \qquad \forall v \in H_0^1(\Omega).$$

Here $\tau := T/N$ and $t_n := n\tau$.

Derive the following stability estimate for any $1 \leq m \leq N$

$$\|u^m\|_{L_2(\Omega)}^2 \leq \|u_0\|_{L_2(\Omega)}^2 + \tau\sum_{n=1}^m \|\theta f(t_n) + (1-\theta)f(t_{n-1})\|_{H^{-1}(\Omega)}^2$$

Hint: Recall that $(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$ and $(a-b)b = \frac{1}{2}a^2 - \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2$.

Problem 3. For this problem, you may use without proof the Denis-Lions and Bramble-Hilbert lemmas as long as you accurately state them.

Let $K = [0, 1]$, $\mathcal{P} = \mathbb{P}^2$ and $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$ where for $q \in \mathbb{P}^2$

$$\mathcal{N}_1(q) = q(0), \qquad \mathcal{N}_2(q) = q(1), \qquad \mathcal{N}_3(q) = \int_0^1 q.$$

(1) Prove or disprove that $(K, \mathcal{P}, \mathcal{N})$ is a finite element triplet.
(2) Find the dual basis of $\mathbb{P}^2$, i.e., $\{\lambda_1, \lambda_2, \lambda_3\}$ such that $\mathcal{N}_j(\lambda_i) = \delta_{ij}$, $1 \le i, j \le 3$.
(3) Define the finite interpolant $I_K : C^0(K) \to \mathcal{P}$ using the previously computed basis.
(4) Show that there is an absolute constant C such that for all $w \in H^3(K)$ there holds

$$\|w - I_K w\|_{L_2(K)} \le C|w|_{H^3(K)}.$$

Problem 4. For $f \in C[0, 1]$, we propose to approximate the solution to the following PDE

$$-u''(x) + u(x) = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

Let $N \in \mathbb{N}$, $h := 1/N$, $x_i := ih$ and $U_i \approx u(x_i)$ given by

$$-\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + U_i = f(x_i),\ i = 1, ..., N-1,\ U_0 = U_N = 0.$$

Show that

$$\max_{i=0,\dots,N} |U_i| \le \max_{i=1,\dots,N} |f(x_i)|.$$

<u>*Hint:*</u> Argue for

$$U_k = \max_{i=1,\dots,N-1} U_i \qquad \text{and} \qquad U_l = \min_{i=1,\dots,N-1} U_i.$$