Complex Analysis Qualifying Examination

August 2011

1. Suppose $u(x, y)$ is a (real-valued) harmonic function on a simply connected domain in $\mathbb{C}$. Show that $u(x, y)$ can be written in the form $f(x + iy) + g(x - iy)$, where f and g are holomorphic functions.

2. An *inversion* is a function on the extended complex numbers of the form $z \mapsto \dfrac{1}{z - z_0}$, where z_0 is some complex constant. Show that the dilation $z \mapsto 4z$ can be obtained by composing three inversions.

3. Determine, with proof, the set of all biholomorphic self-mappings of $\mathbb{C} \setminus \{0\}$, the punctured plane.

4. Suppose f is a continuous function on $\{ z \in \mathbb{C} : |z| \le 1 \}$, the closed unit disk, and f is holomorphic on the open unit disk. Prove that if $f(z)$ is real when $|z| = 1$, then f is a constant function.

5. Suppose that g is a bounded, continuous function on the real axis. Show that the improper integral $\int_0^\infty e^{-zt} g(t)\, dt$ (the Laplace transform) represents a holomorphic function of z in the half-plane where $\operatorname{Re} z > 0$.

6. Use the residue theorem to prove that $\displaystyle\int_0^\infty \frac{x^2}{1 + x^5}\, dx = \frac{\pi/5}{\sin(2\pi/5)}$.

7. Find the general form of an entire function f satisfying the property that
$$\frac{f(w) - f(z)}{w - z} = f'\left(\frac{w + z}{2}\right)$$
for all distinct complex numbers w and z.

8. Let $\{f_n\}_{n=1}^\infty$ be the sequence of iterates of the sine function: namely, $f_1(z) = \sin(z)$, and $f_{n+1}(z) = \sin(f_n(z))$ when $n \ge 1$. Show that this sequence $\{f_n\}$ is not locally bounded in any neighborhood of the origin.

9. Suppose that f is holomorphic on $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$ (the punctured unit disk), and f has no zeroes. Show that there exist an integer m and a function g holomorphic on the punctured disk such that $f(z) = z^m e^{g(z)}$ for all z in the punctured disk.

10. State and prove *one* of the following theorems: the Riemann mapping theorem, Runge's theorem about polynomial approximation, or the Schwarz reflection principle.