Complex Analysis Qualifying Exam, January 2025

$\mathbb{D}$ denotes the unit disc $B(0,1)$.

Problem 1: Find all poles of the function $f(z) = 1/(e^z - 1)$ and determine their orders. Find the first three terms of the Laurent expansion of f at zero.

Problem 2: Compute

$$\int_{Im(z)=1} \frac{e^{iz}}{z^4 - 16}\, dz\ .$$

Problem 3: Set $f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{e^{z \log n}}$.

a) Show that this series converges uniformly on compact subsets of $\{z \in \mathbb{C} \,|\, Re(z) > 1\}$.

b) For the Taylor expansion of f at the point $z_0 = 2$, find (i) the radius of convergence and (ii) a formula (an infinite sum) for the $k-th$ Taylor coefficient.

Problem 4: Let $\Omega \subset \mathbb{C}$ be an open set, $\{f_n\}_{n=1}^{\infty}$ a sequence of analytic functions on Ω that converges to f, uniformly on compact subsets of Ω, with f not identically equal to zero. Suppose $f(a) = 0$ for some $a \in \Omega$, and U is a neighborhood of a. Show: then there is n_0 such that for $n \geq n_0$, f_n has a zero in U.

Problem 5: Does there exist an analytic function f in $\mathbb{D}$ such that $|f(z)| \to \infty$ whenever $|z| \to 1$? Justify your answer.

Problem 6: Let $\mathcal{F}$ be a family of analytic functions in $\mathbb{D}$. Show that $\mathcal{F}$ is normal in $H(\mathbb{D})$ if and only if there is a sequence $\{M_n\}$ of positive numbers with $\limsup_{n\to\infty} (M_n)^{1/n} \leq 1$ such that if $f \in \mathcal{F}$ has Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $|a_n| \leq M_n$ for all n.

Problem 7: Suppose g is analytic in $\mathbb{C}\setminus\{0\}$, maps $\mathbb{C}\setminus\{0\}$ to itself, and is injective. Show that $g(z) = \alpha z$ or $g(z) = \alpha/z$ for some $\alpha \in \mathbb{C}$.

Problem 8: Suppose $f(z)$ is analytic near z_0 and $f(z_0) = f'(z_0) = 0$, while $f''(z_0) \neq 0$. Show that there are two smooth curves γ_1 and γ_2 that pass through z_0, are orthogonal at z_0, and so that f restricted to γ_1 is real and has a minimum at z_0, while f restricted to γ_2 is also real, but has a maximum at z_0. *Hint:* near z_0, $f(z) = g(z)^2$ for some injective analytic function g (why?).

Problem 9: Let $0 < |a| < 1$.

a) Show that the infinite product $f(z) = \Pi_{n=1}^{\infty}(1 - a^n z)$ defines an entire function.
b) Find the order of f.

Problem 10: State (but do not prove) Runge's theorem, the big Picard theorem, and the monodromy theorem.