Complex Analysis Qualifying Examination

August 2008

All problems are worth 10 points.

1) Let
$$f(z) = \frac{1}{z(z-1)(z-2)} .$$
Find the Laurent expansion of f valid in the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

2) Find a one-to-one conformal map of the region $\{z \in \mathbb{C} \mid |z| < 2 \text{ and } |z-1| > 1\}$ onto the upper half plane.

3) Let f and g be zero-free functions on the unit disc. If $f'(1/n)/f(1/n) = g'(1/n)/g(1/n)$ for all $n \in \mathbb{N}$, what can be said about the relation between f and g? Prove your claim.

4) Let $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $r > 0$. Assume that the function $f(z)$ to which it converges has exactly one singular point z_0 on $\{z \in \mathbb{C} \mid |z| = r\}$, and that z_0 is a simple pole. Show that $\lim_{n\to\infty}(a_n/a_{n+1})$ exists and equals z_0.

5) Evaluate
$$\int_0^{\infty} \frac{\log x}{1+x^2}\, dx .$$

6) Fix the real number $\alpha > 1$. Show that the equation
$$\sin z = e^{\alpha} z^3$$
has exactly three solutions in the unit disc.

7) a) Show that there exists exactly one germ $[f]_0$ of a holomorphic function in a neighborhood of 0 such that $f(0) = 0$ and $f(z) = z + (1/2)f(z)^2$.

b) Show that this germ admits unrestricted analytic continuation in $\mathbb{C} \setminus \{\frac{1}{2}\}$.

c) Is there a holomorphic function g on $\mathbb{C} \setminus \{\frac{1}{2}\}$ such that $[g]_0 = [f]_0$? Justify your answer.

8) In the right-hand half plane, suppose $f_0(z) = z$ and for each positive integer n, $f_n(z)$ is the principal branch of $z^{f_{n-1}(z)}$. Is there a neighborhood of the point 1 in which the family $\{f_n\}_{n=1}^{\infty}$ is a normal family? Explain why or why not.

9) Prove that the range of the entire function $z^2 + \cos(z)$ is all of $\mathbb{C}$.

10) Prove that if $K = \{x + iy \in \mathbb{C} \mid |x| \leq 1 \text{ and } |y| \leq 1\}$, and $u(x, y)$ is a harmonic function in a neighborhood of K, then for every $\varepsilon > 0$ there exists a harmonic polynomial $p(x, y)$ such that $\max_{(x,y)\in K} |u(x, y) - p(x, y)| < \varepsilon$.