Complex analysis qualifying exam, January 2009.

1. Give the statements of the following theorems:

(a) Runge's theorem;
(b) the Mittag-Leffler theorem.

2. Let $f(z)$ be analytic in $\{|Re\ z| < 1\}$ and continuous on the closure of that domain. Suppose that $f(z)$ is real on the lines $x = \pm 1$. Prove that then $f(z)$ can be analytically continued to the whole plane and that the resulting entire function satisfies $F(z+4) = F(z)$ for all $z \in \mathbb{C}$.

3. Let u and v be non-constant harmonic functions on a complex domain. Prove that uv is harmonic if and only if $u + icv$ is analytic for some real c. (Hint: one of the possible ways to prove the "only if" part is to consider f/g with $f = u_x - iu_y,\ g = v_x - iv_y$.)

4. Prove that for any $a \in \mathbb{C}$ and any integer $n \geq 2$ the polynomial $1 + z + az^n$ has at least one root in the disk $\{|z| \leq 2\}$. (Hint: use the Vieta theorem that says that the product of the roots of a monic polynomial is equal to its constant term in absolute value.)

5. Let f be an analytic function in the unit disk $\mathbb{D}$ satisfying $0 < |f(z)| < 1$. Show that then for any $z \in \mathbb{D}$

$$|f(z)| \leq |f(0)|^{\frac{1-|z|}{1+|z|}}.$$

(Hint: estimate $\log |f|$.)

6. Calculate the integral using residues:

$$\int_0^\infty \frac{dx}{(x^2+4)x^{1/3}}.$$

7. (a) Let f be a non-constant holomorphic function on a neighborhood of the closed unit disk such that $|f(z)|$ is constant on the unit circle. Prove that f has at least one zero in the unit disk.

(b) Find all entire f such that $|f|$ is constant on the unit circle.

8. Let $f(z)$ be analytic in the strip $\{|Re\ z| < \pi/4\}$ and satisfy $f(0) = 0, |f(z)| < 1$. Prove that then $|f(z)| \leq |\tan z|$ for all z from the strip.

9. Let f be a holomorphic function in the unit disk $\mathbb{D}$ that is injective and satisfies $f(0) = 0$. Prove that there exists a holomorphic function g in $\mathbb{D}$ such that $(g(z))^2 = f(z^2)$ for all $z \in D$.

10. Let f be analytic in a bounded connected domain Ω and continuous in the closure of Ω. Suppose that the boundary of Ω consists of two disjoint smooth simple closed curves γ_1 and γ_2. Prove that f has an analytic antiderivative in Ω if and only if $\int_{\gamma_1} f(z)dz = 0$ (note that $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$).