Real Analysis Qualifying Exam; August, 2009.

Work as many of these ten problems as you can in four hours. Start each problem on a new sheet of paper.

#1. Evaluate the iterated integral

$$\int_0^\infty \int_0^\infty x \exp(-x^2(1+y^2))\,dx\,dy.$$

(Justify your answer.)

#2. Let $f \in C[0,1]$ be real–valued. Prove that there is a monotone increasing sequence of polynomials $\{p_n(x)\}_{n=1}^\infty$ converging uniformly on $[0,1]$ to $f(x)$.

#3. Let $\{f_n\}_{n=1}^\infty$ be a sequence of non-zero elements of $L^2[0,1]$. Prove that there is a function $g \in L^2[0,1]$ such that for all $n \geq 1$ we have

$$\int_0^1 g(x) f_n(x)\,dx \neq 0.$$

#4. Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$. Given sets $A_i \in \Sigma$, $i \geq 1$, prove that

$$\mu\Big(\bigcap_{i=1}^\infty A_i\Big) = \lim_{n\to\infty} \mu\Big(\bigcap_{i=1}^n A_i\Big).$$

Give an example to show that this need not hold when $\mu(X) = \infty$.

#5. Let K be a compact subset of $\mathbf{R}^n$ and describe the dual space of the Banach space $C(K)$. (You may choose either the real or the complex Banach space.)

Let $\mathbf{1} \in C(K)$ denote the constant function taking value 1 and let S be the subset of the dual space consisting of the positive bounded linear functionals on $C(K)$ that map $\mathbf{1}$ to 1. Show that the extreme points of S are the point evaluation maps, $f \mapsto f(x)$.

#6. Let $\ell^2(\mathbf{Z})$ denote the real Hilbert space of square–summable functions on the integers. Let x_k $(k \geq 1)$ be a sequence in $\ell^2(\mathbf{Z})$ that converges coordinate–wise to zero, i.e., such that $\lim_{k\to\infty} x_k(n) = 0$ for all $n \in \mathbf{Z}$.

Must x_k converge in norm to 0 as $k \to \infty$? What about if $\|x_k\|$ is assumed to be bounded?

Must x_k converge weakly to 0 as $k \to \infty$? What about if $\|x_k\|$ is assumed to be bounded?

Justify your answers (by proof or counter-example.)

#7. Let X be a second countable (that is, having a countable basis of open sets) and normal topological space. Show that there is a countable family $\mathcal{F}$ of countinuous functions from X into the interval $[0,1]$ that separates points and closed sets: i.e., such that if $x \in X$ and C is a closed subset of X with $x \notin C$, then there is $f \in \mathcal{F}$ such that $f(x) = 0$ and $f(C) \subseteq \{1\}$.

#8. Let $f \in L^1(0, \infty)$ and define

$$h(x) = \int_0^\infty (x+y)^{-1} f(y)\, dy$$

for $x > 0$. Show that h is differentiable at all $x > 0$ and show $h' \in L^1(r, \infty)$ for every $r > 0$. What about for $r = 0$? (Justify your answer.)

#9. Suppose X is a Banach space and Y is a normed linear space and $T : X \to Y$ is a linear map such that for every bounded linear functional $g \in Y^*$ we have $g \circ T$ is bounded. Show that T is bounded.

#10. Let X be a real Banach space and suppose C is a closed subset of X such that

(i) $x_1 + x_2 \in C$ for all $x_1, x_2 \in C$,
(ii) $\lambda x \in C$ for all $x \in C$ and $\lambda > 0$,
(iii) for all $x \in X$ there exist $x_1, x_2 \in C$ such that $x = x_1 - x_2$.

Prove that, for some $M > 0$, the unit ball of X is contained in the closure of

$$\{x_1 - x_2 \mid x_i \in C,\ \|x_i\| \le M,\ (i = 1, 2)\}.$$

Deduce that, for some $K > 0$, every $x \in X$ can be written $x = x_1 - x_2$, with $x_i \in C$ and $\|x_i\| \le K\|x\|$, $(i = 1, 2)$. (In fact, any $K > M$ will do, but you need not show this.)