Qualifying Examination in Real Variables, August 2010

(1) (a) Give an example of a sequence (f_n) in $L_1[0,1]$ such that $\lim_{n\to\infty} \|f_n\|_{L_1} = 0$, but (f_n) does not converge to 0 almost everywhere.

(b) Show that if a sequence (f_k) in $L_1[0,1]$ satisfies $\|f_k\|_{L_1} \leq 2^{-k}$ for $k \geq 1$, then $f_k \to 0$ almost everywhere.

(2) Let E be a subset of $[0,1]$ with positive outer Lebesgue measure, i.e. $m^*(E) > 0$. Show that for each $\alpha \in (0,1)$ there is an interval $I \subset [0,1]$ so that

$$m^*(E \cap I) \geq \alpha \text{ length}(I).$$

(3) Let X be a Banach space and let (x_n) be a sequence from X that converges weakly to 0. Prove that the sequence $(\|x_n\|)$ is bounded.

(4) (a) Let (f_n) be a bounded sequence in $C[0,1]$. Prove that

$$(f_n) \text{ converges weakly to } 0 \iff (f_n) \text{ converges pointwise to } 0.$$

(b) Assume that $(f_n) \subset C[0,1]$ converges in the weak topology. Show that f_n is norm convergent in $L_1[0,1]$.
[For part (b) you may use problem (3).]

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that for some $C > 0$

$$m\{x : |f(x)| \geq \lambda\} \leq C\lambda^{-2}, \text{ for all } \lambda > 0.$$

Prove that there is some $C' > 0$ so that

$$\int_E |f(x)|dx \leq C'\sqrt{m(E)}, \text{ for all measurable } E \subset \mathbb{R}.$$

(6) Let $f(x)$ be a continuous function on $[0,1]$ with a continuous derivative $f'(x)$. Given $\varepsilon > 0$, prove that there is a polynomial $p(x)$ so that

$$\|f(x) - p(x)\|_\infty + \|f'(x) - p'(x)\|_\infty < \varepsilon.$$

(7) Let X be a non-empty complete metric space and let

$$\{f_n : X \to \mathbb{R}\}_{n=1}^\infty$$

be a sequence of continuous functions with the following property: for each $x \in X$, there exists an integer N_x so that $\{f_n(x)\}_{n\geq N_x}$ is either a monotone increasing or decreasing sequence. Prove that there is a non-empty open subset $U \subseteq X$ and an integer N so that the sequence $\{f_n(x)\}_{n\geq N}$ is monotone for all $x \in U$.

(8) Assume that $1 \leq p < \infty$ and that a linear operator $T : L_p[0,1] \to L_p[0,1]$ is such that (Tf_n) converges almost everywhere to 0 if (f_n) converges almost everywhere to 0.

Show that T is a bounded operator on $L_p[0,1]$.

(9) (a) State the Hahn Banach Theorem for real vector spaces.

(b) Deduce from it the following corollary: Let X be a Banach space, $Y \subset X$ a closed subspace and $x \in X \setminus Y$. Show that there is an $x^* \in X^*$ so that $x^*|_Y \equiv 0$ and $x^*(x) = 1$.

(10) Let U be the closed unit ball in the Banach space $C[0,1]$ of continuous real valued functions on the unit interval. Prove that the extreme points of U are the constant functions ± 1. Prove that $C[0,1]$ is not a dual Banach space.