Real Analysis Qualifying Exam
August 1, 2022

These ten problems are equally weighted. Solve as many problems or portions thereof as you can in four hours. Please start the solution of each problem you attempt on a fresh sheet of paper.

#1. Let $f : (0,1) \to \mathbb{R}$.

(a) Give the definition of *absolute continuity* for f.

(b) Show that if $E \subseteq (0,1)$ has Lebesgue measure 0 and f is monotone and absolutely continuous, then $f(E)$ has Lebesgue measure zero.

#2. Suppose X is a compact Hausdorff space and $f : X \to \mathbb{R}$ is continuous. Let $\epsilon > 0$. Show the existence of an open set $U \subseteq X$ and a continuous function $g : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) \subseteq U$, $g(U) = \{0\}$, and $\|g - f\|_u < \epsilon$, where the norm is the uniform norm.

#3. Let X be a complete metric space that is uncountable but has a countable dense subset D. Suppose $f : X \to X$ satisfies $f(X \setminus D) \subseteq D$ and $f(D) \subseteq X \setminus D$. Show that f cannot be everywhere continuous on X.

#4. Either give an example of a σ-finite measure space $(X, \mathcal{M}, \mu)$ with an uncountable family $(A_\lambda)_{\lambda \in \Lambda}$ of pairwise disjoint measurable sets $A_\lambda \in \mathcal{M}$, each with $\mu(A_\lambda) > 0$, or prove that such an example cannot exist.

#5. Let $\mathcal{V} \subseteq C[0,1]$ be the linear span of the polynomials $\{x^{2n} \mid n \in \mathbf{N},\, n > 0\}$. For which values of p, $1 \le p \le +\infty$, is $\mathcal{V}$ dense in $L^p([0,1])$, (defined using Lebesgue measure on $[0,1]$, of course)? Prove that your answer is correct.

#6. Let $T : X \to Y$ be a bounded linear operator between Banach spaces. Let X^* and Y^*, respectively, denote the dual spaces consisting of bounded linear functionals of X and Y. Let $T^* : Y^* \to X^*$ be defined by $(T^*\phi)(x) = \phi(Tx)$ for $\phi \in Y^*$ and $x \in X$. (You may assume and need not prove that T^* is well defined.)

(a) Show that T^* is linear and bounded and satisfies $\|T^*\| = \|T\|$.

(b) Suppose that T is onto Y and show that there exists $c > 0$ such that $\|T^*\phi\| \ge c\|\phi\|$ for all $\phi \in Y^*$.

#7. Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of all continuous functions from X to $\mathbb{C}$, endowed with the usual uniform norm. Suppose $(f_n)_{n=1}^\infty$ is a bounded sequence in $C(X)$. Show that this sequence converges to 0 in the weak topology on $C(X)$ if and only if it converges pointwise to 0, namely,

$$\forall x \in X \quad \lim_{n\to\infty} f_n(x) = 0.$$

#8. Evaluate the limit
$$\lim_{n\to\infty}\int_E \left(1+\frac{x}{n}\right)^n \frac{e^{-x}}{(x^2-1)}\,d\lambda(x),$$
where λ is Lebesgue measure on $\mathbb{R}$ and $E=[2,\infty)$. Be sure to justify your assertions.

#9. (a) State the Principle of Uniform Boundedness.
(b) Suppose that X and Y are real Banach spaces and that $\Phi : X \times Y \to \mathbb{R}$ is bilinear, meaning that it is linear in each variable separately. Suppose that
  (i) for all $x \in X$ there exists $A_x \geq 0$ such that
$$\forall y \in Y \quad |\Phi(x,y)| \leq A_x\|y\|$$
  (ii) for all $y \in Y$ there exists $B_y \geq 0$ such that
$$\forall x \in X \quad |\Phi(x,y)| \leq B_y\|x\|.$$
Show there is a constant $K \geq 0$ such that
$$\forall x \in X,\, \forall y \in Y \quad |\Phi(x,y)| \leq K\|x\|\,\|y\|.$$

#10. For every natural number n, let $\mathcal{V}_n$ denote the subspace of all polynomials of degree $\leq n$, regarded as a subspace of $C[0,1]$. Let
$$\mathcal{V}_\infty = \bigcup_{n\geq 1}\mathcal{V}_n.$$
For which $n \in \{1,2,\ldots,\infty\}$ does there exist a bounded linear functional ϕ on $C[0,1]$ such that
$$\forall q \in \mathcal{V}_n \quad \phi(q) = q'(1)\ ?$$
Justify your answer.