Real Analysis Qualifying Exam, January, 2017

(1) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Prove directly from the definition of convergence almost everywhere that if for all n,
$\mu\{x \in \Omega : |f_n(x)| > 1/n\} < n^{-3/2}$, then $f_n \to 0$ a.e. (μ).

(2) Find all f in $L^1(1,2)$ such that for every natural number n we have $\int_1^2 x^{2n} f(x)\,dx = 0$. Give reasons for all assertions you make.

(3) A. Prove that there exists a sequence of measurable functions g_n on $[0,1]$ such that:
(a) $g_n(x) \geq 0$ for any $x \in [0,1]$;
(b) $\lim_{n\to\infty} g_n(x) = 0$ a.e.;
(c) For any continuous function $f \in C[0,1]$,

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x)\,dx = \int_0^1 f(x)\,dx.$$

B. If g_n is a sequence of measurable functions on $[0,1]$ such that (a), (b), and (c) are satisfied, what can you say about $\int_0^1 \sup_n g_n(x)\,dx$?

(4) We say that a sequence $\{a_n\}_{n=1}^{\infty}$ in $[0,1]$ is equidistributed (in $[0,1]$) if and only if for all intervals $[c,d] \subset [0,1]$,

$$\lim_{n\to\infty} \frac{|\{\, a_1, \ldots, a_n \,\} \cap [c,d]|}{n} = d - c.$$

(Here $|A|$ denotes the number of elements in the set A.)
Let $\mu_N = \frac{1}{N} \sum_{1 \leq n \leq N} \delta_{a_n}$ with δ_{a_n} the point measure at a_n, that is, for any subset $S \in [0,1]$, $\delta_{a_n}(S) = \begin{cases} 1 & \text{if } a_n \in S \\ 0 & \text{if } a_n \notin S \end{cases}$.
Show that $\{a_n\} \subset [0,1]$ is equidistributed if and only if

$$\lim_{N\to\infty} \int_0^1 f d\mu_N = \int_0^1 f dm,$$

for all continuous functions on $[0,1]$, where m is Lebesgue measure.

(5) Consider the space $C([0,1])$ of real-valued continuous functions on the unit interval $[0,1]$. We denote by $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$ the supremum norm of $f \in C([0,1])$ and by $\|f\|_2 := (\int_0^1 |f(x)|^2 dx)^{\frac{1}{2}}$ the L^2-norm of $f \in C([0,1])$. Let S be a closed linear subspace of $(C([0,1]), \|\cdot\|_\infty)$. Show that if S is complete in the norm $\|\cdot\|_2$, then S is finite-dimensional.

(6) Prove that if a function $f : [0,1] \to \mathbb{R}$ is Lipschitz, with
$$|f(x) - f(y)| \leq M|x - y|$$
for all $x,\ y \in [0,1]$, then there is a sequence of continuously differentiable functions $f_n : [0,1] \to \mathbb{R}$ such that
(i) $|f_n'(x)| \leq M$ for all $x \in [0,1]$;
(ii) $f_n(x) \to f(x)$ for all $x \in [0,1]$.

(7) Given $f : \mathbb{R} \to \mathbb{R}$ bounded and uniformly continuous and K_n with $K_n \in L^1(\mathbb{R})$ for $n = 1, 2, 3, \cdots$ such that
(i) $\|K_n\|_1 \leq M < \infty$, $n = 1, 2, 3, \cdots$.
(ii) $\int_{-\infty}^{\infty} K_n(x)dx \to 1$ as $n \to \infty$.
(iii) $\int_{\{x:|x|>\delta\}} |K_n(x)| \to 0$ as $n \to \infty$ for all $\delta > 0$.
Show that $K_n * f \to f$ uniformly, where
$$K_n * f(x) = \int_{-\infty}^{\infty} K_n(y)f(x-y)dy.$$

(8) A. Construct a Lebesgue measurable subset A of of $\mathbb{R}$ so that for all reals $a < b$, $0 < m(A \cap [a,b]) < b - a$, where m is Lebesgue measure on $\mathbb{R}$.
B. Suppose $A \subseteq \mathbb{R}$ is a Lebesgue measurable set and assume that
$$m(A \cap (a,b)) \leq \frac{b-a}{2}$$
for any $a, b \in \mathbb{R}$, $a < b$. Prove that $\mu(A) = 0$.

(9) Prove or disprove that the unit ball of $L^7(0,1)$ is norm closed in $L^1(0,1)$.

(10) Let C be the Banach space of convergent sequences of real numbers under the supremum norm. Compute the extreme points of the closed unit ball, B, of C and determine whether B is the closed convex hull of its extreme points.

(11) Show that every convex continuous function defined on the closed unit ball of a reflexive Banach space achieves a minimum. (A convex function on a convex subset A of a normed space is a real valued function, f, on A s.t. for every $x, y \in A$ and every $0 < \lambda < 1$ we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.)