Real Analysis Qualifying Exam
Texas A&M University, January 2019

Printed name: ______________________________

The Aggie code of honor:"An Aggie does not lie, cheat or steal or tolerate those who do".

Signed name: ______________________________

Student ID number: ______________________________

Instructions:

- Read problems very carefully. If you have any questions raise your hand.
- Every question is worth 10 points.
- Justify every non trivial step and give proper citations in your proofs.

1. True or false (prove or give a counter example)
   (a) Let $E \subset \mathbb{R}$ be a Borel set, then $\{(x,y) \in \mathbb{R}^2 : x - y \in E\}$ is a Borel set in $\mathbb{R}^2$.
   (b) Let $E \subset Q := [0,1] \times [0,1]$. Assume that for every $x, y \in [0,1]$ the sets $E_x = \{y \in [0,1] : (x,y) \in E\}$ and $E^y = \{x \in [0,1] : (x,y) \in E\}$ are Borel. Then E is Borel.
   (c) A function $f : \mathbb{R} \mapsto \mathbb{R}$ is called Lipschitz if there exists a $\xi > 0$ such that $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| \le \xi |x - y|$. If $A \subset \mathbb{R}$ is Lebesgue measurable and f is Lipschitz then $f(A)$ is Lebesgue measurable.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space. Is it true that for every measurable essentially bounded $f : X \to \mathbb{R}$ we have $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$? Give an answer both in the case that μ is finite and the case μ is σ-finite.

3. Let $f : \mathbb{R} \to \mathbb{R}$ Lebesgue integrable and for $n \in \mathbb{N}$ define
$$g_n(x) = n \int_{(x, x+\frac{1}{n})} f d\lambda.$$
   (a) Prove that $\lim_{n\to\infty} g_n = f$ λ-a.e.
   (b) Pove that for every $n \in \mathbb{N}$, $\int_{\mathbb{R}} |g_n| d\lambda \le \int_{\mathbb{R}} |f| d\lambda$.
   (c) Prove $\lim_{n\to\infty} \int_{\mathbb{R}} |g_n| d\lambda = \int_{\mathbb{R}} |f| d\lambda$.

4. Let $f \in L^1((0,1]^2, \lambda_2)$ such thaty $\int_{(0,x]\times(0,y]} f d\lambda_2 = 0$ for ever $x, y \in (0,1]$. Prove that $f = 0$ λ_2-a.e.

5. Let λ be the Lebesgue measure on $\mathbb{R}$. Let $E \subset \mathbb{R}$ be Lebesgue measurable such that $0 < \lambda(E) < \infty$. Prove that for all $0 \leq \gamma < 1$ there exists an open interval $I \subset \mathbb{R}$ such that

$$\lambda(E \cap I) \geq \gamma \lambda(I).$$

6. Let X be a compact metrizable space and $\{\mu_n\}$ a sequence of Borel measures on X with $\mu_n(X) = 1$ for every n. Consider the linear map $\varphi : C(X) \to \ell^\infty(\mathbb{N})$ defined by $\varphi(f) = (\int_X f \, d\mu_n)_n$. What conditions on the sequence $\{\mu_n\}$ are equivalent to φ being an isometry? Provide justification.

7. Let X be a compact metric space and $\{f_n\}$ a sequence in $C(X)$. Prove that $\{f_n\}$ converges weakly in $C(X)$ if and only if it converges pointwise and $\sup_n \|f_n\| < \infty$. Also, give an example of an X and a sequence $\{f_n\}$ in $C(X)$ which converges weakly but not uniformly.

8. Let X be a Banach space. Show that if X^{**} is separable then so is X. Also, give an example, with justification, to show that the converse is false.

9. (a) Let X be a compact metrizable space. Describe the dual of $C(X)$ according to the Riesz representation theorem.
   (b) Consider the spaces $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and $Y = [0,1]$ with the topologies inherited from $\mathbb{R}$. Prove that there does not exist a bijective bounded linear map from $C(X)$ to $C(Y)$.

10. Let X be a Banach space and Y a subspace of X. Show that $\|x + Y\| = \inf\{\|x + y\| : y \in Y\}$ defines a norm on X/Y if and only if Y is closed.