Qualifying Examination in Real Variables, January 2020

General Instructions:

(1) Use a separate sheet of paper for each problem.

(2) Unless stated otherwise, you may use results from Folland's book, but you need to state them carefully (if you do not remember their names you can state them).

Problems:

(1) Show that

$$\lim_{n\to\infty}\int_0^\infty \frac{4t^3+12}{12t^6+3nt+2}dt = 0.$$

(2) Show for all $f \in L_1(\mathbb{R})$ that

$$\lim_{\delta\to 1}\int |f(\delta x) - f(x)|\, dx = 0.$$

(3) For an integrable function $f \in L_1(\mathbb{R})$, and $\alpha \geq 0$ put

$$E_\alpha = \{x \in \mathbb{R} : |f(x)| \geq \alpha\}.$$

Show that the map $\alpha \mapsto m(E_\alpha)$ is measurable and that

$$\int_{-\infty}^\infty |f(x)|\, dx = \int_0^\infty m(E_\alpha)\, d\alpha.$$

(4) Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Using the inequality

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b, \text{ for } 0 < \lambda < 1 \text{ and } 0 \leq a, b.$$

prove the Hölder inequality.

(5) Show that for $\epsilon > 0$ there is a closed subset $E \subset [0,1]$ with empty interior, of Lebesgue measure at least $1 - \epsilon$.

(6) Let X be a Banach space and Y a non trivial closed subspace of X.

(a) Show that for all $y^* \in Y^*$ (the dual of Y) the set

$$\big\{x^* \in X^* : \|x^*\| = \|y^*\| \text{ and } x^*|_Y = y^*\big\}$$

is weak*-compact.

(b) Show that every extreme point of the closed unit ball of Y^* extends to an extreme point of the unit ball of X^*.

(7) Assume that $(X, \|\cdot\|)$ is a normed linear space and that Y is a subspace of X. Assume that $\|\!|\cdot|\!\|$ is a norm on Y which is equivalent to $\|\cdot\|$. Proof that $\|\!|\cdot|\!\|$ can be extended to an equivalent norm on all of X.

(8) Let (f_n) be a sequence of continuous functions on $[0,1]$, such that for each $x \in [0,1]$ there is an $n_x \in \mathbb{N}$, so that $f_n(x) \geq 0$ for all $n \geq n_x$.

Show that there are an $N \in \mathbb{N}$ and an open nonempty interval $I \subset [0,1]$, so that $f_n(x) \geq 0$ for all $n \geq N$ and $x \in I$.

(9) For a bounded sequence $(f_n) \subset C[0,1]$, show that

$$f_n \to_{n\to\infty} 0, \text{ weakly} \iff f_n(x) \to_{n\to\infty} 0 \text{ for all } x \in [0,1].$$

(10) On the set $[0,\infty]$ consider the topology $\mathcal{T}$ generated by the open sets (in the usual topology) of $[0,\infty)$ and the sets of the form $[0,\infty] \setminus C$, with $C \subset [0,\infty)$ compact.

(a) Show that $[0,\infty]$ with above defined topology is a compact space.

(b) Show that $[0,\infty]$ with above defined topology is metrizable. Hint: consider a continuous, strictly increasing, and bounded function $f : [0,\infty) \to [0,\infty)$.

(c) Show that the linear space generated by the functions of the form e^{-nx^2}, $n = 1,2,3\ldots$, is dense (with respect to sup-norm) in the space of all continuous functions $f : [0,\infty] \to \mathbb{R}$, having the property that $f(\infty) = 0$.