QUALIFYING EXAM-REAL ANALYSIS
JANUARY 2023

The 10 problems below are equally weighted. Solve as many problems or portions thereof as you can in 4 hours. Please start the solution of each problem you attempt on a new sheet in your bluebook.

In the sequel, unless specified otherwise, $\mathbb{R}$ (or a subset of it) is always equipped with the Borel σ-algebra and the Lebesgue measure (denoted by λ).

Problem 1.
Show that there exists a constant $c > 0$ (and give its value) so that for every measurable function $f: \mathbb{R} \to [0, \infty)$ we have

$$\int_{\mathbb{R}} f^4 d\lambda = c \int_{[0,\infty)} t^3 \lambda(\{f \geqslant t\}) d\lambda(t).$$

Problem 2.
Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions from X to $\mathbb{R}$ such that $\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$ for some integrable function $f: X \to \mathbb{R}$. Show that for all $\varepsilon > 0$ there is $A \in \mathcal{M}$ satisfying $\mu(A) < \infty$ and for all $n \geqslant 1$,

$$\int_{X\setminus A} |f_n| d\mu < \varepsilon.$$

Problem 3.
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from $[0, 1]$ to $\mathbb{R}$.

(1) Show that if $\lim_{n\to\infty} \int_{[0,1]} |f_n - f| d\lambda = 0$ for some integrable function $f: [0, 1] \to \mathbb{R}$, then $(f_n)_{n\in\mathbb{N}}$ converges in λ-measure to f.
(2) Show that if $(f_n)_{n\in\mathbb{N}}$ converges λ-almost everywhere towards a measurable function $f: [0, 1] \to \mathbb{R}$, then $(f_n)_{n\in\mathbb{N}}$ converges in λ-measure to f.
(3) Does the conclusion in assertion (2) still hold if the functions are defined on $\mathbb{R}$ instead?

Problem 4.
Recall that a collection $\mathscr{F}$ of measurable functions from $[0, 1]$ to $\mathbb{R}$ is said to be uniformly integrable *if*

$$\lim_{\lambda(A)\to 0} \sup_{f\in\mathscr{F}} \int_A |f| d\lambda = 0.$$

(1) Given a non-negative $g \in L_1([0, 1])$, show that $\mathscr{F}_g \stackrel{\text{def}}{=} \{f \in L_1([0, 1]) : |f| \leqslant g\}$ is uniformly integrable.
(2) Show that the closed unit ball of $L_2([0, 1])$ is a uniformly integrable subset of $L_1([0, 1])$.

Problem 5.

(1) *Show that a compact metric space is separable.*
(2) *Prove or disprove that the unit ball of ℓ_∞ equipped with the norm topology is separable.*
(3) *Prove or disprove that the unit ball of ℓ_∞ equipped with the weak*-topology is separable.*

Problem 6.

Consider the Banach space $C[0,1]$ consisting of all continuous, real valued functions on $[0,1]$, endowed with the uniform norm, $\|\cdot\|_\infty$. For $f \in C[0,1]$, let

$$\|f\|_L = |f(0)| + \sup_{0\leqslant x<y\leqslant 1} \frac{|f(y)-f(x)|}{y-x}$$

(1) *Show that $\{f \in C[0,1] \mid \|f\|_L \leqslant 1\}$ is compact in $C[0,1]$.*
(2) *Is the set $\{f \in C[0,1] \mid \|f\|_L < \infty\}$ dense in $C[0,1]$ or not? Justify your answer.*

Problem 7.

Suppose X is a real Banach space and $Y \subseteq X$ is a proper subspace. Show that the following are equivalent:

(1) *For every $z \in X$ such that $z \notin Y$, there exists a bounded linear functional ϕ on X such that $\phi(z) = 1$ and, for all $y \in Y$, $\phi(y) = 0$.*
(2) *Y is closed in X.*

Problem 8.

(1) *Let X be a normed vector space and Y be a subspace of X. Show that if Y has non-empty interior then $Y = X$.*
(2) *Let X be a Banach space and T be a bounded operator on X. Show that if for all $x \in X$, there exists $n \in \mathbb{N}$ such that $T^n(x) = 0$, then there exists $d \in \mathbb{N}$ such that for all $x \in X$, $T^d(x) = 0$.*

Problem 9.

Let $(X, \|\cdot\|)$ be a normed vector space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be weakly Cauchy *if for all $x^* \in X^*$, $(x^*(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.*

(1) *Show that a weakly Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X is bounded.*
(2) *Show that for every weakly Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X, there exists $x^{**} \in X^{**}$ such that $(x_n)_{n\in\mathbb{N}}$ weak*-converges to x^{**} and $\|x^{**}\| \leqslant \liminf_{n\to\infty} \|x_n\|$.*

Problem 10.

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-negative continuous functions on $[0,1]$ such that for each $k \in \mathbb{N} \cup \{0\}$, the limit $\lim_{n\to\infty} \int_{[0,1]} t^k g_n(t) d\lambda(t)$ exists. Show that there exists a unique finite positive Radon measure μ on $[0,1]$ such that for all continuous functions on $[0,1]$, $\int_{[0,1]} f d\mu = \lim_{n\to\infty} \int_{[0,1]} f(t) g_n(t) d\lambda(t)$.