REAL ANALYSIS QUALIFYING EXAM - JANUARY 2025

The 10 problems below are equally weighted (parts of probems are **not** necessarily equally weighted). Please start the solution of each problem you attempt on a new sheet. Make sure to properly mention any named theorem that you will need in any of your solutions.

In this exam, m, dx, and dy all denote the Lebesgue measure.

(1) Let f be a nonnegative Lebesgue integrable function on a measurable subset $A \subset \mathbb{R}$. Define $\phi(t)$ on $(0, \infty)$ by

$$\phi(t) = m(\{x \in A : f(x) > t\}).$$

Show that $\int_A f = \int_0^\infty \phi$.

(2) Construct a function $f \in L^1([0,1])$ such that f is essentially unbounded on any open interval (c, d) for $0 \leq c < d \leq 1$. (Recall that a function on a measure space with underlying set X is said to be essentially unbounded if it is unbounded on $X \setminus N$ for any measure 0 set $N \subset X$.)

(3) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function in two variables, such that $x \mapsto f(x, y)$ is a measurable function for each y. Suppose that for each $(x, y) \in [0,1] \times [0,1]$, $\frac{\partial f}{\partial y}$ exists and that there is a real valued function $g \in L^1([0,1])$ such that $\left|\frac{\partial f}{\partial y}(x, y)\right| \leq g(x)$ for any $(x, y) \in [0,1] \times [0,1]$. Show that

$$\frac{d}{dy} \int_0^1 f(x, y) dx = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx$$

for any $y \in [0,1]$.

(4) Construct a sequence of functions $\{f_n\}$ in $L^1([0,1])$ such that $f_n(x) \to 0$ for almost every x and for any continuous function $g : [0,1] \to \mathbb{R}$, $\int_{[0,1]} f_n(x) g(x) dx \to \int_{[0,1]} g(x) dx$, or show such a sequence does not exist.

(5) let f_n be a sequence of measurable functions on $\mathbb{R}$ such that $|f_n(x)| \leq 1$ for all x and such that $f_n \to 0$ almost everywhere in $\mathbb{R}$. Let $g \in L^1(\mathbb{R})$ and consider the sequence of functions

$$g * f_n(x) = \int g(x - y) f_n(y) dy.$$

(a) Show that for any x, $g * f_n(x) \to 0$ as $n \to \infty$.
(b) Show that for any $M > 0$, $g * f_n(x) \to 0$ uniformly on $[-M, M]$ as $n \to \infty$.

(6) Let $\{f_n\}$ be a bounded sequence in $L^2([0,1])$. Suppose that $\{f_n\}$ converges almost everywhere. Show that $\{f_n\}$ weakly converges in $L^2([0,1])$.

(7) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space $\mathcal{X}$ such that $\|\cdot\|_1 \leq \|\cdot\|_2$. Show that if $\mathcal{X}$ is complete with respect to both norms, then the norms are equivalent.

(8) Let X be a Banach space and $Y \subset X$ be a closed subspace. Let X/Y be the quotient space as vector spaces. Endow X/Y with a norm as follows: for $[x] \in X/Y$

$$||[x]|| = \inf\{||y|| : y \in [x]\}.$$

Prove that X/Y is a Banach space. (You do **not** have to show that the norm on X/Y is a norm.)

(9) (a) State Alaoglu's Theorem.
(b) A topological space is said to be sequentially compact if every sequence in the space has a convergent subsequence. Construct a compact space which is not sequentially compact.

(10) (a) Let $f : [0,1] \to \mathbb{R}$ be a function. Let

$$D = \{x \in [0,1] : f \text{ is not continuous at } x\}.$$

Prove that D is a countable union of closed sets.
Hint: Define $\omega(f, x) = \inf_{h>0}\{\sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in (x - h, x + h) \cap [0,1]\}\}$. Show that for any $c > 0$, $\{x : \omega(f, x) \geq c\}$ is a closed set.
(b) Prove there is no function $f : [0,1] \to \mathbb{R}$ such that f is continuous at any rational number but not continuous at any irrational number.