Topology-Geometry Qualifying Examination
August 2011

Instructions. Answer all questions. Write your name and page number in the upper right corner of each page. Start each problem on a new sheet of paper, and use only one side of each sheet.

Notation. $\mathbb{R}$ denotes the real numbers, and $\mathbb{Q}$ denotes the rational numbers regarded as a subspace of $\mathbb{R}$. $\mathbb{R}^n$ denotes Euclidean n-dimensional space. $\mathbb{C}$ denotes the complex numbers and $\mathbb{C}^n$ is complex n-dimensional space.

1. Let $\mathbb{R}^3$ have coordinates (x, y, z), let C be a smooth curve in the plane $z = 0$ and let M be the cylinder over C in $\mathbb{R}^3$. Find the Gauss and mean curvature functions of M in terms of the curvature function of C. (Hint: if C is parameterized by t, parameterize M by (t, z).)

2. Let $\mathbb{R}^2$ have its standard inner product $\langle \cdot, \cdot \rangle$ and let

$$G = \{f : \mathbb{R}^2 \to \mathbb{R}^2 \mid f \text{ is linear and } \langle f(v), f(w) \rangle = \langle v, w \rangle \; \forall v, w \in \mathbb{R}^2\}.$$

(a) Show that G is a Lie group.

(b) Show that the tangent bundle to the connected component of the identity in G is diffeomorphic to $S^1 \times \mathbb{R}$, where S^1 denotes the circle.

3. Let $f(x, y, z), g(x, y, z)$ be smooth functions on $\mathbb{R}^3$ and let $X = f(x, y, z)\frac{\partial}{\partial x} + g(x, y, z)\frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$.

(a) Compute the Lie derivative of X with respect to Y at $(0, 0, 0)$.

(b) Give necessary and sufficient conditions on the functions f, g such that there exists a smooth surface through $(0, 0, 0)$ in some neighborhood of $(0, 0, 0)$ of $\mathbb{R}^3$ whose tangent space is spanned by X, Y.

4. Let M be a connected, compact, n-dimensional manifold, let TM denote its tangent bundle and let $\Lambda^n TM$ denote its n-th exterior power.

(a) Show that $\Lambda^n TM$ is a vector bundle and compute its rank.

(b) Give a necessary and sufficient condition on M so that $\Lambda^n TM$ admits a nowhere zero smooth section.

5. Let X be a Hausdorff topological space. Let $f : X \to X$ be a continuous function such that $f \circ f = f$. Prove that $f(X)$ is closed. (Hint. Assume not. Construct an appropriate net that violates uniqueness of limits.)

6. Let X be a Hausdorff topological space and $A \subset X$. Prove that the following three statements are equivalent.

(I) For every $x \in A$, x has a neighborhood $U \subset X$ such that $A \cap U$ is closed in U.

(II) A can be written as the intersection of a closed set and an open set.

(III) $\overline{A} - A$ is closed.

7. Let $\mathbb{C}[z_1, \ldots, z_n]$ denote the space of polynomials in $z_1, \ldots, z_n$ with complex coefficients. For $T \subset \mathbb{C}[z_1, \ldots, z_n]$, define the zero set of T to be

$$Z(T) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in T\}.$$

A subset $U \subset \mathbb{C}^n$ is said to be *Zariski open* if there is a (possibly empty) set $T \subset \mathbb{C}[z_1, \ldots, z_n]$ such that $U = \mathbb{C}^n - Z(T)$.

(a) Let $T_1, T_2 \subset \mathbb{C}[z_1, \ldots, z_n]$. Show that $Z(T_1) \cap Z(T_2) = Z(T_1 \cup T_2)$, and $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$ where $T_1 T_2 = \{fg \mid f \in T_1, g \in T_2\}$.

(b) Prove that the Zariski open sets give a topology (called the *Zariski topology*) on $\mathbb{C}^n$.

(c) Define that a topological space has the *finite complement topology*. Show that the Zariski topology on $\mathbb{C}^1 (= \mathbb{R}^2)$ coincides with the finite complement topology.

8. Let X be a topological space.

(a) Define *locally finite* family of (non-empty) subsets of X. Show that if X is countably compact, then every locally finite family of nonempty subsets of X is finite. (Recall that a space is *countably compact* if and only if every countable cover has a finite subcover.)

(b) Define *paracompact space*. Prove that if X is paracompact and countably compact, then X is compact.

