# Topology/Geometry Qualifying Exam

(August 2012)

**Instructions:**

i. You must work on every problem and prove your assertions.
ii. Start each problem on a separate sheet of paper.
iii. Once you finish the exam, assemble your answers according to the problem numbers.

---

(1) Let $Y$ be a set and consider two collections of functions

$$\mathcal{F} = \{f_\alpha \colon X_\alpha \to Y \mid \alpha \in \Lambda\} \quad \text{and} \quad \mathcal{G} = \{g_\alpha \colon Y \to X_\alpha \mid \alpha \in \Lambda\},$$

where the $X_\alpha$'s are topological spaces.

(a) Show that $Y$ has a unique finest topology $Y_\mathcal{F}$ so that $f_\alpha \in \mathcal{F}$ is continuous for all $\alpha \in \Lambda$. Show that a function $F \colon Y_\mathcal{F} \to Z$ is continuous if and only if $F \circ f_\alpha$ is continuous for all $\alpha \in \Lambda$.

(b) Show that $Y$ has a unique coarsest topology $Y_\mathcal{G}$ so that $g_\alpha \in \mathcal{G}$ is continuous for all $\alpha \in \Lambda$. Show that a function $G \colon Z \to Y_\mathcal{G}$ is continuous if and only if $g_\alpha \circ G$ is continuous for each $\alpha \in \Lambda$.

(2) Given $\alpha \in \mathbb{R}$ denote $L_\alpha = \{(r, \alpha r) \mid r \in \mathbb{Q}\} \subset \mathbb{R}^2$, where $\mathbb{Q}$ denotes the rational numbers. Define $S_{irr} = \cup_{\alpha \in \mathbb{R} - \mathbb{Q}} L_\alpha$ and give $X = \mathbb{R}^2 - S_{irr} \subset \mathbb{R}^2$ the subspace topology.

(a) Show that $X$ is connected and locally path-connected.

(b) Is $X$ paracompact? Explain.

(c) Is $X$ locally compact? Explain.

(3) Let $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$ and $B^{n+1} = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| \leq 1\}$ denote the unit sphere and unit closed ball in $\mathbb{R}^n$, respectively. Show that one can define an equivalence relation $\sim$ on $X = S^n \times [0, \infty)$ so that the quotient space $X/\sim$ is homeomorphic to the ball $B^{n+1}$.

(4) Suppose that $X$ is a $T_1$ topological space which is also *normal*, and that $X = U \cup V$, where $U$ and $V$ are open in $X$. Show that one can find open subsets $U_1, V_1$ satisfying

$$\overline{U_1} \subset U, \quad \overline{V_1} \subset V \quad \text{and} \quad X = U_1 \cup V_1.$$

(5) Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of congruence classes of integers mod $n$, endowed with the discrete topology, and give the cartesian product $X = \prod_{n \geq 2} \mathbb{Z}/n\mathbb{Z}$ the product topology. Denote by $[x]_n$ the congruence class of $x \in \mathbb{Z}$ modulo $n$.

(a) Fix $k \in \mathbb{Z}$ and let $F_k \subset X$ denote the set of elements $\mathbf{x} = ([x_n]_n)_{n \geq 2} \in X$ such that $[x_n]_n$ is a multiple of $[k]_n$ for all $n > k$. Show that $F_k$ is a closed subset of $X$.

(b) Let $B \subset X$ be a non-empty closed subset such that $B \cap F_2 = \emptyset$. Show that there is a continuous function $f \colon X \to [0, 1]$ such that $f(x) = 0$ if $x \in B$ and $f(x) = 1$ if $x \in F_2$.

(6) Let $X \subset \mathbb{C}^n$ denote the subspace given by the equations

$$z_1^2 + z_2^2 + \cdots + z_n^2 = 0 \quad \text{and} \quad |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 2,$$

where $|z|$ denotes the norm of a complex number. Let $Gr_2(\mathbb{R}^n)$ denote the Grassman manifold of 2-planes in $\mathbb{R}^n$.

(a) Show that $X$ is a smooth, compact submanifold of $\mathbb{C}^n$ and determine its dimension.

(b) Write $\mathbf{z} = \mathbf{x} + \sqrt{-1}\ \mathbf{y} \in \mathbb{C}^n$, where $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$ are the real and imaginary parts of $\mathbf{z}$, respectively. Show that the map $\psi \colon X \to Gr_2(\mathbb{R}^n)$ sending $\mathbf{z} \in X$ defined by $\psi(\mathbf{z}) = Span_{\mathbb{R}}(\mathbf{x}, \mathbf{y})$ is a smooth, surjective map.

- *Here, $Span_{\mathbb{R}}(\mathbf{x}, \mathbf{y})$ denotes the linear span in $\mathbb{R}^n$ of the vectors $\mathbf{x}$ and $\mathbf{y}$.*

(7) Let $M$ be a smooth manifold and let $\alpha$ be a smooth section of its cotangent bundle. For $x \in M$, let

$$\alpha_x{}^{\perp} := \{v \in T_xM \mid \alpha(v) = 0\}$$

and let $\alpha^{\perp} := \cup_{x \in M} \alpha_x{}^{\perp}$. Show that $\alpha^{\perp}$ is a sub-vector bundle of the tangent bundle $TM$ if and only if $\alpha$ is a non-vanishing section of the cotangent bundle. You may use the fact that $TM$ is a vector bundle.

(8) Let $M$ be a smooth manifold and let $\alpha \in \Omega^1(M)$ be a non-vanishing section. Consider the following statements:

(a) There exists a function $f \in C^{\infty}(M)$, such that $\alpha = df$.

(b) Through each $x \in M$ there exists a hypersurface $Z_x \subset M$, such that $\alpha^{\perp} |_{Z_x} = TZ_x$.

(c) For all $X, Y \in \Gamma(\alpha^{\perp})$, i.e., $X, Y$ are sections of the vector bundle $\alpha^{\perp}$, $[X, Y] = 0$.

(d) For all $X, Y \in \Gamma(\alpha^{\perp})$, $[X, Y] \in \Gamma(\alpha^{\perp})$.

Determine the implications among them (e.g. $(x)$ implies $(y)$ because ...).

(9) Let $\mathbb{R}^3$ have coordinates $(x, y, z)$. Which of the following are Riemannian metrics on $\mathbb{R}^3$:

(a) $g = (x + y)dx \circ dx + (y + z)dy \circ dy + (z + x)dz \circ dz$.

(b) $g = 13dx \circ dx + 2dx \circ dy + 44dy \circ dy + dz \circ dz$.

(c) $g = dx \circ dx + dy \circ dy$.

Here $\circ$ denotes the symmetric tensor product.

(10) Compute the Gauss and mean curvature functions for a sphere of radius 5 in Euclidean three space.