TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
AUGUST 2015

INSTRUCTIONS:.

- There are 8 problems. Work on all of them.
- Prove your assertions.
- Use a separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Problem 1. *Let X be the interval $[0,1]$ with the following topology. A subset U of X is open if and only if it contains the interval $(0,1)$ or it does not contain the point $1/2$.*

(a) *Is the topology on X smaller (coarser) than, larger (finer) than, or not comparable to the the standard topology on the unit interval? Please justify your answer.*
(b) *Determine the closure of the set $\{1/4\}$ in X. Please justify your answer.*
(c) *Show that X is a T_0 space, but it is not a T_1 space.*

Problem 2. *Let X be a compact space, $\{C_j \mid j \in J\}$ a nonempty family of closed sets in X, $C = \bigcap_{j\in J} C_j$, and U an open set in X containing C. Show that there exists a finite subset $\{j_1, j_2, \ldots, j_n\}$ of J such that*

$$C_{j_1} \cap C_{j_2} \cap \cdots \cap C_{j_n} \subseteq U.$$

Problem 3. *Let X and Y be topological spaces, and $f : X \to Y$ and $g : Y \to X$ be two maps such that, for all $y \in Y$, $f(g(y)) = y$. Show that if Y is connected and $f^{-1}(y)$ is connected for all $y \in Y$, then X is connected.*

Problem 4. *Let (X, d) be a compact metric space and $f : X \to X$ be a distance preserving map (a map such that, for all $x, y \in X$, $d(f(x), f(y)) = d(x, y)$) .*
(a) *Show that f is injective.*
(b) *Show that, for every point $x \in X$ and every ε-ball $B_\varepsilon(x)$ centered at x, one of the balls in the sequence*

$$f(B_\varepsilon(x)),\ f(f(B_\varepsilon(x))),\ f(f(f(B_\varepsilon(x)))),\ \ldots$$

has nonempty intersection with $B_\varepsilon(x)$.
(c) *Use part (b), or any other method, to prove that f is surjective.*

Problem 5. *Let V be a real vector space of dimension $n+1$. Define an equivalence relation on $V\setminus\{0\}$ by $u \sim v$ if $u = \lambda v$ for some nonzero $\lambda \in \mathbb{R}$. Let $\mathbb{P}(V) = (V\setminus\{0\})/\sim$ denote the quotient space, equipped with the quotient topology. Prove that $\mathbb{P}(V)$ is a smooth manifold of dimension n.*

Problem 6. *Let $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half-plane. Let $u \cdot v$ denote the dot product of vectors $u, v \in \mathbb{R}^2$. Use the natural identification $T_{(x,y)}M \simeq \mathbb{R}^2$ to define a metric g on M by*

$$g_{(x,y)}(u, v) \; := \; \frac{u \cdot v}{y^2} \quad \text{for all } \; u, v \in T_{(x,y)}M\,.$$

Compute the Gauss curvature of M.

Problem 7. *Prove that the distribution $\mathcal{D}$ on $\mathbb{R}^3$ spanned by the vector fields*

$$\begin{aligned} X &= (1 + z^2)\frac{\partial}{\partial z}\,, \\ Y &= \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 4(y - x)\frac{\partial}{\partial z} \end{aligned}$$

is involutive. Find flat coordinates for the distribution; that is, find coordinates (u, v, w) on $\mathbb{R}^3$ so that $\mathcal{D}$ is spanned by $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$.

Problem 8. *For what values of $c \in \mathbb{R}$ is $\{xyz = c\} \subset \mathbb{R}^3$ a smooth, embedded submanifold? What are the dimensions of these manifolds?*