TEXAS A&M UNIVERSITY

TOPOLOGY/GEOMETRY QUALIFYING EXAM

August 2019

- `There are 10 problems. Work on all of them and prove your assertions.`
- `Use a separate sheet of paper for each problem and write only on one side of the paper.`
- `Write your name on the top right corner of each page.`

1. Let X be a compact metric space. Show that if $f : X \longrightarrow X$ satisfies $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$ (i.e., if f is an isometry) then f is a homeomorphism.

2. Prove that the metric space X is complete if and only if for every sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ of nonempty closed subsets of X such that diameters of A_n converge to 0, the intersection $\bigcap_{i=1}^{\infty} A_i$ is non-empty.

3. Let X_i, for $i \in I$, be a family of topological spaces, and let $A_i \subset X_i$ be subsets. Show that $\overline{\prod_{i\in I} A_i} = \prod_{i\in I} \overline{A_i}$, where closure on the left-hand side of the equality is taken with respect to the product topology on $\prod_{i\in I} X_i$.

4. Let X and Y be topological spaces, where Y is compact. Let $p : X \times Y \longrightarrow X$ be the projection onto the first factor. Show that p is closed (i.e., maps each closed subset of $X \times Y$ to a closed subset of X).

5. Show that any map $f : S^1 \longrightarrow S^1$ of degree 1 is homotopic to the identity.

6. (a) Given a differential p-form ω on a manifold N and a smooth map $g : M \to N$ give the definition of the pull-back $g^*\omega$ of the form ω by the map g.

   (b) Define $g : \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\} \to \mathbb{R}^3\backslash\{0\}$ by $(x, y, z) = g(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$ and

   $$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

   Compute $g^*\omega$ and $d\omega$ and verify by direct computations that $g^*(d\omega) = d(g^*\omega)$

   (c) Using the calculations of $g^*\omega$ from the previous item, calculate $\int_S \omega$, where S is the upper unit hemisphere in $\mathbb{R}^3$, i.e. $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$.

7. (a) Given a smooth map $F : M \to N$ between two smooth manifolds M and N define the notions of a critical point and a critical value of F.

   (b) Define $\mathcal{Z} := \{(x, p, q) \in \mathbb{R}^3 : x^3 + px + q = 0\}$.

      i. Prove that $\mathcal{Z}$ is a smooth submanifold of $\mathbb{R}^3$;

      ii. Define $\pi : \mathcal{Z} \to \mathbb{R}^2$ by $\pi(x, p, q) = (p, q)$ for every $(x, p, q) \in \mathcal{Z}$. Prove that (p, q) is a critical value of π if and only of $4p^3 + 27q^2 = 0$.

8. Let $H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half plane with the Riemannian metric $g = \dfrac{dx^2 + dy^2}{y^2}$. Calculate the Gaussian curvature of this metric.

9. (a) Let S be a smooth tensor field of type (r, s) on a smooth manifold M and X be a smooth vector field on M. Give the defintion of the Lie derivative $L_X S$ of the tensor field S with respect to the vector field X (Here the definition, which uses certain limit and does not involve Lie brackets, is expected).

   (b) Prove that if X and Y are two smooth vector fields on M, then $L_X Y = [X, Y]$, where $[X, Y]$ is the Lie bracket (the commutator) of X and Y.

   (c) Assume that vector fields X and Y commute and linearly independent in a neighborhood of point p_0 in M, i.e., $[X, Y](p) = 0$ and the dimension of $\text{span}\Big(X(p), Y(p)\Big)$ is equal to 2 for every p in this neighborhood. Prove that there is a coordinate system $(U, x_1, \ldots, x_n)$ around p_0 (here $n = \dim M$) such that $X = \frac{\partial}{\partial x_1}$ and $Y = \frac{\partial}{\partial x_2}$ on U.

10. (a) Assume that $(\omega_1, \ldots \omega_k)$ is a collection of independent 1-forms defining the distribution D in an open set U of M, i.e. $D(p) = \{X \in T_pM : \omega_1(X) = \ldots = \omega_k(X) = 0\}$ for any $p \in U$. Describe the involutivity of D in terms of the forms ω_i.

   (b) Let G be a Lie group and $\mathfrak{g}$ be the corresponding Lie algebra. Recall that the Maurer-Cartan form Ω on G is the $\mathfrak{g}$-valued 1-form satisfying $\Omega_g(v) = (L_{g^{-1}})_* v$ for every $g \in G$ and $v \in T_gG$, where L_g denotes the left translation by g in G. Prove that Ω satisfies

$$d\Omega(X, Y) = -[\Omega(X), \Omega(Y)],$$

   where in the right-hand side $[\cdot, \cdot]$ means the brackets in the Lie algebra $\mathfrak{g}$.

   (c) Here we use the notations of the previous item. Let M be a smooth manifold endowed with a $\mathfrak{g}$-valued 1-form Φ satisfying $d\Phi(X, Y) + [\Phi(X), \Phi(Y)] = 0$. Prove that for any $p \in M$ there exists a neighborhood U of p and a smooth map $F : U \to G$ such that $\Phi = F^*\Omega$. (Hint: Consider an appropriate involutive distribution on $M \times G$ such that the graph of the required map F is an integral submanifold of this distribution).