TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
August 2022

- There are 10 problems. Work on all of them and prove your assertions.
- Use a separate sheet for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

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1. Suppose that (X, d) is a compact metric space.
   (a) Let f be a continuous function on X. Prove that f is uniformly continuous on X.
   (b) Let $C(X) = \{f \mid f : X \to \mathbb{R} \text{ is coutinuous}\}$. The supremum norm on $C(X)$ is given by
   $$\|f\| = \sup_{x \in X} |f(x)|.$$
   Prove that $(C(X), \| \ \|)$ is a **complete** norm space.
   (c) Given two metrics space (X_1, d_1) and (X_2, d_2) that are homeomorphic to each other, if (X_1, d_1) is complete, does it imply (X_2, d_2) is also complete?
2. Suppose that (X, d) is a metric space and T is a map from X to itself and
   $$d(T(x), T(y)) < \lambda d(x, y), 0 < \lambda \leq 1.$$
   for any $x, y \in X$.
   (a) If X is a compact metric space, prove that T has a fixed point;
   (b) If X is a noncompact complete metric space, prove that T has a fixed point if $\lambda < 1$.
3. Let X be a topological space.
   (a) If X is path connected, is X connected? Prove or disprove.
   (b) If X is connected, is X path connected? Prove or disprove.
4. Let $\mathbb{R}P^2$ be the real projective space and $T^2 = S^1 \times S^1$ the torus. Prove that any continuous map $f : \mathbb{R}P^2 \to T^2$ is null-homopotic.
5. Suppose that X is a path connected topological space whose universal cover is compact. Prove that $\pi_1(X)$ is finite.
6. (a) Formulate the Implicit Function theorem.

(b) Let p and q are nonnegative integers such that $p+q=n$ and $I_{p,q}$ be the diagonal $n\times n$ matrix with (i,i) entry equal to 1 if $1\le i\le p$ and to -1 if $p+1\le i\le n$. Prove that the set $O_{p,q}$ of all $n\times n$ matrices A such that $A^T I_{p,q} A = I_{p,q}$ is an embedded submanifold of the space of all $n\times n$-matrices ($\cong \mathrm{R}^{n^2}$). What is the dimension of this submanifold?

(c) Describe explicitly the tangent space to $O_{p,q}$ at the identity.

7. (a) Let $f: M\to N$ be a smooth map and ω be a differential p-form on N. Define the pullback $f^*\omega$.

(b) Given a differential p-form ω and a vector field X on a manifold M give the definition of the interior product $i_X\omega$.

(c) Given a vector field X and a differential p-form ω on a manifold M give the definition of the Lie derivative $L_X\omega$ of ω with respect to the vector field X.

(d) Let M be connected and $\pi: M\times N\to N$ be the natural projection. Prove that a differential form ω on $M\times N$ is equal to $\pi^*\alpha$ for some differential p-form on N if and only if and only if $i_X\omega=0$ and $L_X\omega=0$ for every vector field X on $M\times N$ such that $d\pi(X(m,n))=0$ at each point $(m,n)\in M\times N$.

8. (a) Given a Lie group G with the identity e describe how to endow the space T_eG with the structure of a Lie algebra.

(b) Prove that for a matrix Lie group (i.e. for a Lie subgroup of $GL(n,\mathbb{R})$) the operation of Lie brackets on its Lie algebra coincides with the matrix commutator.

(c) Let $E_n^+(\mathbb{R})$ be the group of the rigid motions of $\mathbb{R}^n$, i.e. the group of transformation of the form $x\mapsto a+Ax$, $a\in\mathbb{R}^n$, and $A\in SO_n(\mathbb{R})$ with operation of composition. Describe the Lie algebra of $E_n^+(\mathbb{R})$.

Hint: Embed $E_n^+(\mathbb{R})$ into $\mathrm{GL}(n+1,\mathbb{R})$.

9. Let $\mathrm{Gr}_2(\mathbb{R}^4)$ be the set of all 2-dimensional subspaces in $\mathbb{R}^4$. Describe as completely as possible how to endow $\mathrm{Gr}_2(\mathbb{R}^4)$ with the structure of smooth manifold.

10. (a) Give the definitions of an involutive distribution and of an integral submanifold of a distribution in terms of a local basis of vector fields.

(b) Give at least one characterization of an involutve distribution in terms of defining forms.

(c) Let M and N be 2-dimensional smooth manifolds. Let (α_1,α_2) be a coframe on M and (ω_1,ω_2) be a coframe on N such that there are constants k_1 and k_2 so that

$$d\alpha_1 = k_1\,\alpha_1\wedge\alpha_2, \qquad d\alpha_2 = k_2\,\alpha_1\wedge\alpha_2, \qquad d\omega_1 = k_1\,\omega_1\wedge\omega_2, \qquad d\omega_2 = k_2\,\omega_1\wedge\omega_2.$$

Prove that for every $q\in M$ and $p\in N$ there exist a neighborhood U of q in M, a neighborhood V of p in N, and a diffeomorphism $F: U\to V$ such that $\alpha_1 = F^*\omega_1,\quad \alpha_2 = F^*\omega_2$. in U.

Hint: Let $\pi: M\times N\to M$ and $p: M\times N\to N$ are canonical projections. Work with the distribution on $M\times N$ defined by 1-forms $\pi^*\alpha_1 - p^*\omega_1$ and $\pi^*\alpha_2 - p^*\omega_2$.