Geometry/Topology Qualifying Exam **Summer 2024**

Texas A&M University **Thursday, 8 August 2024**

- Justify all your assertions.
- There are 10 problems. Try to solve all of them and make solutions and proofs as complete as possible.
- Use a separate sheet for each problem.
- Write your name on the top right corner of each page.

Problem 1.

a. Let X, Y be topological spaces. Suppose X is compact and Y is Hausdorff. Show that every continuous bijection $f\colon X \to Y$ is a homeomorphism.

b. Give an example of two topological spaces X, Y, and a continuous map $f\colon X \to Y$ such that f is a bijection but not a homeomorphism.

Problem 2. Let τ be the collection of subsets of $\mathbb{R}^2$ defined as follows: $U \in \tau$ if and only if $U = \emptyset$ or $\mathbb{R}^2 \setminus U$ consists of a (possibly empty) finite union of points and straight lines.

a. Show that τ is a topology on $\mathbb{R}^2$.

b. Determine whether τ is Hausdorff.

c. Show that τ is coarser than the Euclidean topology on $\mathbb{R}^2$.

d. Determine whether $\mathbb{R} \times \{0\}$ is a compact subspace of $(\mathbb{R}^2, \tau)$.

Problem 3. Let X be the set of points in $\mathbb{R}^2$ with at least one irrational coordinate, that is, $X = \mathbb{R}^2 \setminus \mathbb{Q}^2$. Equip X with the subspace topology induced by the Euclidean topology on $\mathbb{R}^2$.

a. Show that X is path-connected.

b. Show that the fundamental group of X is uncountable.

Problem 4. Let $\mathbb{RP}^n$ denote the n-dimensional real projective space, and let $\mathbb{T}^m = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{m\text{-times}}$ denote the m-dimensional torus.

a. Let $n \geq 2$. Show that every continuous map $f\colon \mathbb{RP}^n \to \mathbb{T}^m$ is null-homotopic.

b. Is every continuous map $f\colon \mathbb{RP}^1 \to \mathbb{T}^m$ null-homotopic?

Problem 5. Let $\mathcal{C}$ and ℓ be respectively a circle and a straight line in $\mathbb{R}^3$ with $\mathcal{C} \cap \ell = \emptyset$. Let S be the union of $\mathcal{C}$ and ℓ, that is, $S = \mathcal{C} \cup \ell$. Compute the fundamental group of $\mathbb{R}^3 \setminus S$ in the following two cases:

a. There is a plane Π in $\mathbb{R}^3$ such that ℓ and $\mathcal{C}$ lie in different components of $\mathbb{R}^3 \setminus \Pi$.

b. There is no plane Π in $\mathbb{R}^3$ such that ℓ and $\mathcal{C}$ lie in different components of $\mathbb{R}^3 \setminus \Pi$.

Problem 6. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ denote the 2-torus, defined as the set of points w, x, y, z such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on $\mathbb{S}^1$. Compute $\displaystyle\int_{\mathbb{T}^2} \sigma$, where σ is the following 2-form on $\mathbb{R}^4$:

$$\sigma = xy^2 z\, dw \wedge dy.$$

Problem 7.

a. Given a smooth map $F : M \to N$ between two smooth manifolds M and N define the notions of a critical point and a critical value of F.

b. Define $\mathcal{Z} := \{(x, a, b, c) \in \mathbb{R}^3 : ax^2 + bx + c = 0, a \neq 0\}$.

   i) Prove that $\mathcal{Z}$ is a smooth submanifold of $\mathbb{R}^3$;

   ii) Define $\pi : \mathcal{Z} \to \mathbb{R}^3$ by $\pi(x, a, b, c) = (a, b, c)$ for every $(x, a, b, c) \in \mathcal{Z}$. Prove that (a, b, c) is a critical value of π if and only of $b^2 - 4ac = 0$.

Problem 8. Consider the set of all ordered triples of vectors in $\mathbb{R}^4$ such that they form an orthonormal basis of their linear span. Endow this set with the natural structure of an embedded submanifold in $\mathbb{R}^N$ for certain N. What is the dimension of this submanifold?

Problem 9. Given $g_1, g_2 \in C^\infty(\mathbb{R}^2)$ and two smooth vector fields X_1 and X_2 in $\mathbb{R}^2$ such that X_1 and X_2 are linearly independent at every point (i.e., form a frame) and $[X_1, X_2] = \alpha_1 X_1 + \alpha_2 X_2$ for some smooth functions α_1 and α_2 in $\mathbb{R}^2$, prove that the following system of equations with respect to the function $u \in C^\infty(\mathbb{R}^2)$

$$\begin{cases} X_1(u) = g_1, \\ X_2(u) = g_2. \end{cases}$$

has a solution on $\mathbb{R}^2$ if and only if

$$X_1(g_2) - X_2(g_1) = \alpha_1 g_1 + \alpha_2 g_2.$$

Problem 10. Let D be a distribution on a manifold M, X be a vector field on M, and e^{tX} denote the local flow of X.

a. Prove that if $(e^{tX})_* D = D$ for sufficiently small t then

$$[X, Y] \in D, \quad \forall Y \in D, \tag{1}$$

i.e. the vector field $[X, Y]$ is tangent to D for every vector field Y tangent to D.

b. A vector field X is called an *infinitesimal symmetry* of the distribution D if the relation (1) holds. Assume that $\dim M = n$, $\operatorname{rank} D = m$. Prove that X is an infinitesimal symmetry of D if and only if for every tuple of the locally defining 1- forms $\omega_1, \ldots \omega_{n-m}$ of D the following identities hold:

$$(L_X \omega_i) \wedge \omega_1 \wedge \ldots \omega_{n-m} = 0, \quad \forall i = 1, \ldots, n - m.$$

c. Let $M = \mathbb{R}^3$ with standard coordinates (x, y, z), and $D = \ker \theta$

$$\theta = dz - \frac{1}{2}(xdy - ydx).$$

Prove that for any $f \in C^\infty(\mathbb{R}^3)$ there exists the unique infinitesimal symmetry X_f of D such that $\theta(X_f) \equiv f$. Express X_f explicitly in terms of f.