

-
- Justify all your assertions.
 - There are 10 problems. Try to solve all of them and make solutions and proofs as complete as possible.
 - Use a separate sheet for each problem.
 - Write your name on the top right corner of each page.
-

1. Let X and Y be topological spaces, and let $\pi_X: X \times Y \rightarrow X$ be the projection on the first coordinate, that is, $\pi_X(x, y) = x$ for $(x, y) \in X \times Y$. Prove or disprove the following assertions:
 - (a) π_X is a continuous map.
 - (b) π_X is an open map.
 - (c) π_X is a closed map.
 - (d) π_X is a quotient map.
2. The branching line B is the topological space obtained as the quotient space of $\mathbb{R} \times \{0, 1\}$ with respect to the equivalence relation $(x, 0) \sim (x, 1)$ if and only if $x < 0$. Prove or disprove the following assertions:
 - (a) B is path-connected.
 - (b) B is locally compact, that is, every point has a neighborhood which is itself contained in a compact set.
 - (c) B is Hausdorff.
 - (d) B is a T_1 space, that is, for every pair of distinct points p and $q \in B$, there exist a neighborhood U_p of p and a neighborhood U_q of q such that $q \notin U_p$ and $p \notin U_q$.
 - (e) B is second-countable.
3. Let (X, d) be a metric space, and let Y be a non-empty subset of X . Let $f: X \rightarrow \mathbb{R}_{\geq 0}$ be the distance function from Y , that is,

$$f(x) = \inf \{d(x, y) \mid y \in Y\}.$$

Show that $f(x) = 0$ if and only if $x \in \bar{Y}$, where \bar{Y} denotes the closure of Y .

4. Let $p: E \rightarrow B$ be a covering space. Fix a basepoint $b_0 \in B$, and suppose $p^{-1}(b_0)$ has k elements.
 - (a) Assume B is connected. Show that $p^{-1}(b)$ has also k elements, for every $b \in B$. Prove the assertion under the assumption that B is path-connected to get half points.
 - (b) Assume B is compact. Show that E is also compact.
5. (a) Compute the fundamental group of the 2-sphere with k points removed.

- (b) Let ℓ_1, \dots, ℓ_n be n distinct lines in \mathbb{R}^3 passing through the origin. Let L be the union of these lines, that is, $L = \bigcup_{i=1}^n \ell_i$. Compute the fundamental group of $\mathbb{R}^3 \setminus L$.
6. (a) Formulate the implicit function theorem (you do not have to prove it).
 (b) Let n be a positive integer and let $O(n)$ denote the set of orthogonal $n \times n$ matrices as a subset of the set of all $n \times n$ matrices $M(n, n)$ (which can be identified with the Euclidean space \mathbb{R}^{n^2}). Prove that $O(n)$ is an embedded submanifold of $M(n, n)$ and find its dimension.
7. Let M and N be smooth manifolds and let $f : M \rightarrow N$ be a smooth map.
 (a) Define the map $f^* : \Omega^k(N) \rightarrow \Omega^k(M)$ that pulls k -forms on N back to k -forms on M .
 (b) For a 1-form $\omega \in \Omega^1(N)$, show that

$$d(f^*\omega) = f^*(d\omega).$$

8. Consider the plane \mathbb{R}^2 (with coordinates (x, y)) equipped with the metric

$$\frac{4}{(1 + x^2 + y^2)^2} (dx^2 + dy^2).$$

Find the Gaussian curvature of this metric at each point.

9. Equip the Euclidean space \mathbb{R}^3 with cylindrical coordinates (r, θ, z) (so that $x = r \cos \theta$, $y = r \sin \theta$, $z = z$). Let Δ be the distribution spanned by X and Y , where

$$X = \frac{\partial}{\partial r}, \quad \text{and} \quad Y = \frac{\partial}{\partial \theta} - r^2 \frac{\partial}{\partial z}.$$

Is the distribution Δ integrable?

10. Let ω be a closed 1-form (so $d\omega = 0$) on a smooth manifold M . Prove that ω is exact (so $\omega = df$ for some smooth function f on M) if and only if

$$\int_{\gamma} \omega = 0$$

for every smooth closed curve γ on M .