TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
January 2025

- There are 10 problems. Work on all of them and prove your assertions.
- Use a separate sheet for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Topology Part:

Problem 1 Let U be an open subset of a topological space. Is it true that U equals the interior of its closure? Justify your answer.

Problem 2 Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.
(b) Let X' denote the a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

Problem 3 Let X be a locally compact Hausdorff space. Is it true that if X has a countable basis, then X is metrizable? Prove your answer.

Problem 4 Compute the fundamental group of the closed orientable surface of genus 2.

Problem 5 Let $\mathbb{S}^n$ be the n-dimensional sphere.

(a) Show that $\mathbb{S}^1$ and $\mathbb{S}^n$ are not homeomorphic if $n > 1$.
(b) Show that $\mathbb{S}^2$ and $\mathbb{S}^n$ are not homeomorphic if $n > 2$.

Differential Geometry part:

Problem 6 Let M be a smooth manifold and $\alpha_1, \cdots, \alpha_k$ are closed differential forms on M, i.e, $d\alpha_i = 0$ for all $i = 1, \cdots, k$.

(a) Prove that $\alpha_1 \wedge \cdots \wedge \alpha_k$ is closed.
(b) If one of $\alpha_1, \ldots, \alpha_k$, say α_1, is an exact differential form, i.e. there exists a differential form β such that $d\beta = \alpha_1$, prove that $\alpha_1 \wedge \ldots \wedge \alpha_k$ is exact.

Problem 7

(a) Are the following subsets embedded submanifolds of $\mathbb{R}^2$:
  (i) $x^3 - y^3 = 0$,
  (ii) $x^2 - y^2 = 0$,
  (iii) $x^3 - y^2 = 0$?
  Prove your answer in each case.
(b) Show that the set of $n \times n$ matrices of rank 1 is an embedded submanifold of the space of all $n \times n$ matrices. What is the dimension of this submanifold?

Problem 8

(a) Given a differential 1-form ω and two vectors fields X_1 and X_2 on a manifold M let

$$T(X_1, X_2) = X_1\big(\omega(X_2)\big) - X_2(\big(\omega(X_1)\big) - \omega([X_1, X_2]).$$

Prove that for every $f_1, f_2 \in C^\infty(M)$

$$T(f_1X_1, f_2X_2) = f_1f_2T(X_1, X_2).$$

(b) Let $\mathrm{Vec}(M)$ be a set of all smooth vector fields on a manifold M [1]. Assume that $\omega : \mathrm{Vec}(M) \to C^\infty(M)$ is a $C^\infty(M)$-linear map, i.e.

$$\omega(f_1X_1 + f_2X_2) = f_1\omega(X_1) + f_2\omega(X_2), \quad \forall X_1, X_2 \in \mathrm{Vec}(M), f_1, f_2 \in C^\infty(M).$$

Prove that if X_1 and X_2 are two vector fields such that for some point $p \in M$ we have $X_1(p) = X_2(p)$ then

$$\omega(X_1)(p) = \omega(X_2)(p),$$

i.e. ω defines a differential 1-form on M.

Problem 9 Let M be a 2-dimensional Riemannian manifold Let (X_1, X_2) be a local orthonormal frame defined in a neighborhood $U \subset M$. Let (ω^1, ω^2) be the dual coframe to (X_1, X_2), i.e. ω^i, $i = 1, 2$, are 1-forms such that for every $\mathbf{x} \in U$ we have $\omega^i|_{\mathbf{x}}(X_j(\mathbf{x})) = \delta_i^j$.

(a) Prove that there exists the unique 1-form ω_2^1 such that

$$d\omega^1 = -\omega_2^1 \wedge \omega^2, \qquad d\omega^2 = \omega_2^1 \wedge \omega^1$$

The form ω_2^1 is called the *canonical connection form* of the orthomormal frame (X_1, X_2).

(b) Let $(\tilde{X}_1, \tilde{X}_2)$ be another local orthomormal frame in U and $(\widetilde{\omega}^1, \widetilde{\omega}^2)$ be its dual coframe. Assume for simplicity that $(\tilde{X}_1, \tilde{X}_2)$ defines the same orientation on U as (X_1, X_2). Let $\widetilde{\omega}_2^1$ be the canonical connection form of the frame $(\widetilde{X}_1, \widetilde{X}_2)$. Prove that

$$d\widetilde{\omega}_2^1 = d\omega_2^1.$$

(c) Let K and $\widetilde{K}$ are to functions on $\mathbb{R}^2$ such that

$$d\omega_2^1 = K\omega^1 \wedge \omega^2, \quad d\widetilde{\omega}_2^1 = \widetilde{K}\widetilde{\omega}^1 \wedge \widetilde{\omega}^2.$$

Prove that $\widetilde{K} \equiv K$.

Problem 10

(a) Give the definitions of an involutive distribution and of an integral submanifold of a distribution in terms of a local basis of vector fields.

(b) Let M and N be 2-dimensional smooth manifolds. Let (X_1, X_2) be a frame on M and (Y_1, Y_2) be a frame on N such that there are constants k_1 and k_2 so that

$$[X_1, X_2] = k_1X_1 + k_2X_2, \qquad [Y_1, Y_2] = k_1Y_1 + k_2Y_2.$$

Prove that for every $q \in M$ and $p \in N$ there exist a neighborhood U of q in M, a neighborhood V of p in N, and a diffeomorphism $F : U \to V$ such that

$$Y_1 = F_*X_1, \quad Y_2 = F_*X_2.$$

Hint: Define a special involutive distribution on on the manifold $M \times N$.

[1] In fact, $\mathrm{Vec}(M)$ is a $C^\infty(M)$–module.