GEOMETRY/TOPOLOGY QUALIFYING EXAM

August 2008

INSTRUCTIONS:

- You must work on all problems below.
- Use a separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Problem 1. Let $\mathbf{R}_\ell$ be the real line $\mathbf{R}$ with the lower limit topology, generated by half-open intervals $[x, y)$ (also called the Sorgenfrey line). Recall that a space is called Lindelöf if every open cover has a countable subcover.

a. Prove that $\mathbf{R}_\ell$ is first countable, Lindelöf, and separable, but not second countable.

b. Let $\mathbf{R}_\ell^2$ be the plane $\mathbf{R}^2$ with the product topology $\mathbf{R}_\ell \times \mathbf{R}_\ell$. Show that $\mathbf{R}_\ell^2$ is first countable and separable, but not Lindelöf.

Problem 2. In this problem all spaces are assumed to be T_1.

a. Fix a subbasis $\mathcal{S}$ of X. Prove that X is completely regular ($T_{3\frac{1}{2}}$) if and only if for each point $x \in X$ and neighborhood $V \in \mathcal{S}$ with $x \in V$, there exists a map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X - V$.

b. Show that the arbitrary product of completely regular spaces is completely regular. (Hint: Use a.)

c. Show that any subspace of a completely regular space is completely regular.

Problem 3. Recall that $g \colon X \to Y$ is called a *proper map* if $g^{-1}(C)$ is compact whenever $C \subset Y$ is compact. Show that if a map $f \colon X \to Y$ is closed and $f^{-1}(y)$ is compact for all $y \in Y$, then f is proper.

Problem 4. A *topological manifold* M is a Hausdorff and second countable space which is locally homeomorphic to an open subset of an Euclidean space. Show that a topological manifold is metrizable. Is M paracompact? Is M normal?

Problem 5. Given $\mathbf{u} \in \mathbf{R}^n$ and $c \in \mathbf{R}$, define

$$S := \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^n \times \mathbf{R}^m \mid \langle \mathbf{x}, \mathbf{u} \rangle^2 = \|\mathbf{y}\|^2 \ + \ c\},$$

where $\langle \mathbf{x}, \mathbf{u} \rangle$ is the inner product of $\mathbf{x}$ and $\mathbf{u}$, and $\|\mathbf{y}\|$ is the norm of $\mathbf{y}$. For which constants c is S a smooth submanifold of $\mathbf{R}^n \times \mathbf{R}^m$? Prove your assertion.

Problem 6. Let X and Y be smooth manifolds.

a. Show that $T_{(x,y)}(X \times Y) = T_x X \times T_y Y$.

b. Define the tangent bundle $T(X)$ of X. Describe how $T(X)$ is given the structure of a smooth manifold.

Problem 7. Classify all surfaces with both Gauss curvature and mean curvature equal to zero. (Provide a proof.)

Problem 8. Consider the surface of revolution obtained by rotating the curve

$$t \mapsto (0, \cos t, \sin t)\,, \quad 0 < t < \pi/2$$

about the line $y = 2$. Identify the image of the Gauss map.

Problem 9. Let S be a compact surface in $\mathbf{R}^3$.

a. Show that S has a point of positive Gauss curvature.
b. Prove or give a counterexample: *S cannot be a minimal surface.*

Problem 10. True or false (answer must be justified): A differential 1-form ω defined on $M := \mathbf{R}^2 \backslash (1, 0)$ such that $d\omega = 0$ and for which there does not exist a smooth function $f : M \to \mathbf{R}$ with $df = \omega$ cannot be extended smoothly to define a differential form on $\mathbf{R}^2$.