Texas A&M Mathematics Department Mini Fair Math Contest

Grades 8-12 Solutions

April 16, 2011

1. What is the sum of the first 100 even numbers.

Solution. Using that the sum of the first 100 even numbers is twice the sum of the first 100 numbers, and the formula

$$\frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n,$$

we conclude that the sum is 10100.

2. Which statement is true about $\left(\frac{10}{11}\right)^{111} \cdot \left(\frac{11}{10}\right)^{211}$?

a. The product is greater than 1,000.
b. The product is greater than 700 but less than 1000.
c. The product is greater than 3 but less than 700.
d. The product is greater than 1 but less than 3.
e. The product is less than 1 but greater than 0.

Solution. We have

$$\left(\frac{10}{11}\right)^{111} \cdot \left(\frac{11}{10}\right)^{211} = (1.1)^{100} = (1.21)^{50} = \left((1.21)^5\right)^{10}.$$

Using that $(1.21)^5 > (1.2)^4 > 2$, we conclude

$$\left(\frac{10}{11}\right)^{111} \cdot \left(\frac{11}{10}\right)^{211} = (1.1)^{100} = \left((1.21)^5\right)^{10} > 2^{10} = 1024 > 1000.$$

3. Solve the equation $|x-3| + |x-6| + |x-9| = 12$.

Solution. There are two values for x which satisfy the above equation: $\{2, 10\}$. One can get these by solving $3 - x + 6 - x + 9 - x = 12$ and $x - 3 + x - 6 + x - 9 = 12$. Because $f(x) = |x-3| + |x-6| + |x-9| - 12$ is a convex function it can intersect the x-axis at most two times which implies that these are the only solutions.

4. Evaluate the product $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)$.

Solution. We have $(\log_2 3)(\log_3 4) = \log_2 4$ and $(\log_2 4)\log_4 5) = \log_2 5$. Applying this strategy two more times we have $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8) = \log_2 8 = 3$

5. Evaluate $\sec(\frac{5\pi}{12})$.

Solution. We have

$$\sec(5\pi/12) = \frac{1}{\cos(\pi/6 + \pi/4)} = \frac{1}{\cos(\pi/6)\cos(\pi/4) - \sin(\pi/6)\sin(\pi/4))} = \sqrt{6} + \sqrt{2}.$$

6. In a triangle the side lengths are 12, 16, and 18. Is the triangle acute, right, or obtuse?

Solution. Since $12^2 + 16^2 > 18^2$ the triangle is acute.

7. Find $\tan C$, where C is the angle opposite to side c of a triangle whose side lengths a, b, and c satisfy

$$\frac{a^3+b^3+c^3}{a+b+c}=c^2.$$

Solution. Multiplying the above identity by $a+b+c$, we obtain $a^3+b^3=(a+b)c^2$. Solving for c, we find $c^2=a^2+b^2-ab$. From the law of cosines we have $\cos C=1/2$. Then we obtain $\sin C=\frac{\sqrt{3}}{2}$ and $\tan C=\sqrt{3}$.

8. Given the points $A=(1,1.5)$ and $B=(4,5)$ find the minimum distance $|AC|+|CB|$ where C is a point on the line $y=1-2x$.

Solution. Let A' be the reflection of A about the line $y=1-2x$. It is easy to verify that $A'=(-1,\frac{1}{2})$. Given any point C on the line $2x+y=1$, we have

$$|AC|+|BC|=|A'C|+|BC|\geq|A'B|$$

because $A'BC$ is a triangle and the sum of any two sides in a triangle is greater or equal to the third side. Note that equality is possible only when C is the intersection point of the segment $A'B$ and the line $2x+y=1$. In any case, we have that $|A'B|$ is the minimum distance and

$$|A'B|=\sqrt{25+\frac{81}{4}}=\frac{\sqrt{181}}{2}.$$

9. Find the number of zeros at the end of 2011!

Solution. We need to find the power of 5 that divides 2011!. The way we count this is the following: we have $\left[\frac{2011}{5}\right]$ numbers less than 2011 divisible by 5, we also have $\left[\frac{2011}{5^2}\right]$ numbers less than 2011 divisible by 25, and so on... Note that the number $\frac{2011}{5^5}$ is less than one. Therefore, we derive that power of 5 that divides 2011! is

$$\sum_{k=1}^{\infty}\left[\frac{2011}{5^k}\right]=402+80+16+3=501.$$

Similarly, we can find the power of 2 that divides 2011! but that number is clearly bigger. Therefore, the number of zeros at the end of 2011! is 501.

10. Simplify the expression

$$\sqrt{6+\sqrt{11}}-\sqrt{6-\sqrt{11}}.$$

Solution. Let $x=\sqrt{6+\sqrt{11}}-\sqrt{6-\sqrt{11}}$. Then we have

$$x^2=6+\sqrt{11}-2\sqrt{6+\sqrt{11}}\sqrt{6-\sqrt{11}}+6-\sqrt{11}=12-10=2$$

and we conclude $x=\sqrt{2}$.

11. If $x+y=16$, find $\max(\min\{x,y\})$.

Solution. Without loss of generality, let $y\leq x$. Then

$$\min\{x,y\}=y\leq\frac{x+y}{2}=8$$

and we derive $\max(\min\{x,y\})\leq 8$. Thus, the maximum value is not bigger than 8. We have that $\max(\min\{8,8\})=8$ and we conclude $\max(\min\{x,y\})=8$.

12. An integer is chosen at random from the set $\{x \,|\, 1 < x \leq 501\}$. Find the probability that this integer is divisible by 7 or 11.

Solution. There are 500 possible integers in the set. Among them there are 71 multiples of 7 and 45 multiples of 11, and 6 multiples of $77 = 7 \times 11$. Hence, there are $110 = 71 + 45 - 6$ multiples of 7 or 11. Then the probability is $110/500 = 11/50$.

13. If $\log_y x + \log_x y = 8$, then find the value of $(\log_y x)^2 + (\log_x y)^2$.

Solution. Let $z = \log_y x$. Then the problem reduces to computing the value of $z^2 + \dfrac{1}{z^2}$ given that $z + \dfrac{1}{z} = 8$. Then, we derive

$$z^2 + \frac{1}{z^2} = \left(z + \frac{1}{z}\right)^2 - 2 = 8^2 - 2 = 62.$$

14. Let $f(x)$ be a function such that $f(x) + 2f(-x) = \cos x$ for every real number x. What is the value of $f(\pi)$?

Solution. We use $x = \pi$ and obtain $f(\pi) + 2f(-\pi) = \cos \pi = -1$. Similarly, we get $f(-\pi) + 2f(\pi) = -1$ when $x = -\pi$. Eliminating $f(-\pi)$ from the two equations, we obtain $f(\pi) - 4f(\pi) = -1 + 2 = 1$. Hence, we have $f(\pi) = -\frac{1}{3}$.

15. Solve the equation $2^x - 2^{-x} = \sqrt{5}$.

Solution. If we denote $a = 2^x$, we have that $a^2 - \sqrt{5}a - 1 = 0$. The only positive solution is $a = \frac{\sqrt{5}+3}{2}$ which gives $x = \log_2 \frac{\sqrt{5}+3}{2}$.

16. There are ten different science books on the same shelf. Four of them are about mathematics and the rest are physics books. Find the probability of the event that all mathematical books are next to each other and all physics books are next to each other on the shelf.

Solution. The number of all possible placements is 10!. The number of placements with all of the math books next to each other and all of the physics books next to each other is $2 \cdot 4! \cdot 6!$. Therefore the probability p is

$$p = \frac{2 \cdot 4! \cdot 6!}{10!} = \frac{1}{105}.$$

17. Determine the number of solutions of the equation $\cos(x) = \dfrac{x}{2011}$ on the interval $[\pi, 16\pi]$.

Solution. Note that the graph of $y = \cos(x)$ can intersect the graph of $y = \frac{x}{2011}$ only once on each of the intervals $[k\pi, (k+1)\pi]$ for any $k = 1, 12, \ldots, 15$. Then the equation has exactly 15 solutions on the interval $[\pi, 16\pi]$.

18. Find all positive solutions of the system of equations

$$xy = 12\sqrt{6}$$
$$yz = 54\sqrt{2}$$
$$zx = 48\sqrt{3}.$$

Solution. We multiply the equations and obtain $(xyz)^2 = 12 \cdot 54 \cdot 48 \cdot 6 = 432^2$. Then $xyz = 432$ and using $xy = 12\sqrt{6}$ we obtain $z = 6\sqrt{6}$. Similarly, we get $x = 4\sqrt{2}$ and $y = 3\sqrt{3}$. Then the solution is $(x, y, z) = (4\sqrt{2}, 3\sqrt{3}, 6\sqrt{6})$

19. Find the limit of the sequence $\sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \ldots$.

Solution: Note that the sequence a_n satisfies $a_{n+1} = \sqrt{1+a_n}$. It is easy to verify that the sequence a_n is monotone increasing and $\sqrt{2} \leq a_n \leq \sqrt{3}$. Therefore, it has a limit which we denote with α. Then α satisfies $\sqrt{2} \leq \alpha \leq \sqrt{3}$ and the equation $\alpha = \sqrt{1+\alpha}$. Solving the equation, we obtain $\alpha = \frac{1+\sqrt{5}}{2}$.

20. Solve the equation

$$(x+1)(x+2)(x+3)(x+4)+1=0.$$

Solution. The equation can be written as

$$(x^2+5x+4)(x^2+5x+6)+1=0.$$

Let $y = x^2+5x$. Then we have $(y+4)(y+5)+1 = (y+5)^2$. This implies $y=-5$ and solving $-5 = x^2+5x$ we obtain $x = (-5-\sqrt{5})/2$ and $x = (-5+\sqrt{5})/2$ to be the two double roots of the equation.