

NUMERICAL ANALYSIS QUALIFIER

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Do four out of the following five problems. Note that you may use the Poincaré inequality (when appropriate) without proving it.

Problem 1. (a) Consider the inner product

$$\langle f, g \rangle = \int_{-1}^1 (x+1)f(x)g(x) dx$$

and let Π_n be the set of polynomials of degree at most n . Let $Q \in \Pi_n$ be a nontrivial polynomial which is $\langle \cdot, \cdot \rangle$ -orthogonal to Π_{n-1} . Consider the quadrature

$$I_1(g) = \sum_{i=1}^n c_i g(x_i) \approx \int_{-1}^1 (x+1)g(x) dx.$$

where $\{x_i\}$ is the set of zeros of Q and c_i are chosen so that $I_1(g)$ is exact for Π_{n-1} . Show that $I_1(g)$ is, in fact, exact for Π_{2n-1} .

(b) Show that there exist coefficients $\{a_i\}$ for $i = 0, \dots, n$ which make the following formula exact for Π_{2n} .

$$I(f) = a_0 f(-1) + \sum_{i=1}^n a_i f(x_i) \approx \int_{-1}^1 f(x) dx.$$

Here the nodes x_i are given in Part (a).

(c) Show that the quadrature of Part (b) is not exact for Π_{2n+1} .

Problem 2. Let A be a symmetric and positive definite real $n \times n$ matrix. Let $\{k_i\}$ be a sequence of integers satisfying $0 = k_0 < k_1 < \dots < k_l = n$ and consider the block form of A which results when the indices are partitioned into the sets $(k_0 + 1, k_1), \dots, (k_{l-1} + 1, k_l)$, i.e.,

$$(2.1) \quad A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1l} \\ A_{21} & A_{22} & \cdots & A_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l1} & A_{l2} & \cdots & A_{ll} \end{pmatrix}$$

where the block A_{ij} is the corresponding $(k_i - k_{i-1}) \times (k_j - k_{j-1})$ submatrix of A . Consider the block SOR method for iteratively computing the solution of $Ax = b$ given by

$$(2.2) \quad (\omega^{-1}D + L)x_{m+1} = ((\omega^{-1} - 1)D - U)x_m + b, \quad \omega \neq 0.$$

Here D consists of the block diagonal matrix with diagonal blocks $\{A_{ii}\}$, $i = 1, \dots, l$ and U and L respectively denote the upper and lower block triangular matrices made from the remaining blocks of A (excluding the diagonal blocks).

- (a) Show that D is invertible.
- (b) Show that $(\omega^{-1}D + L)$ is nonsingular by giving a forward block substitution algorithm for evaluating its inverse applied to a vector $v \in R^n$.
- (c) Show that a necessary condition for the convergence of (2.2) for all starting iterates is that $0 < \omega < 2$.

Problem 3. Consider the implicit Runge Kutta method based on Simpson's rule:

$$\begin{aligned} x_{n+1} &= x_n + \frac{h}{6}\{F_1 + 4F_2 + F_3\} \quad \text{where} \\ F_1 &= f(t_n, x_n), \quad F_3 = f(t_{n+1}, x_{n+1}), \\ F_2 &= f\left(t_{n+1/2}, \frac{x_{n+1} + x_n}{2} - \frac{h}{8}(F_3 - F_1)\right) \end{aligned}$$

for approximating the solution of the initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0.$$

Here $h > 0$ is the step size, $t_n = nh$ and x_n is the approximation to $x(t_n)$.

- Use the truncation error estimate for Simpson's rule to estimate the local truncation error for the above scheme (you need not derive the truncation estimate for Simpson's rule). What is the order of approximation for the above scheme?
- Show that the region of absolute stability for the above scheme contains the negative real axis.

Problem 4. Consider the boundary value problem for the unknown function $u(x)$, $x \in (0, 1)$, given by

$$-\epsilon u'' + bu' + u = f, \quad \text{for } x \in (0, 1), \quad u(0) = 0, \quad u(1) = 0.$$

Here ϵ is a positive constant and $b = b(x)$ and $f = f(x)$ are given smooth functions.

- Derive a weak formulation of the above problem and show that if $1 - b'(x)/2 \geq c_0$ holds for some positive constant c_0 , for all $x \in (0, 1)$, then the corresponding bilinear form is coercive in $H_0^1(0, 1)$.
- Assume that the interval $(0, 1)$ is partitioned into N equal subintervals. Consider the finite element space of continuous piecewise linear functions over this partition. Define the Galerkin finite element approximation to the solution of this problem.
- Give a modification of the above scheme (by changing the contributions of the lower order terms) which results in the global (stiffness) matrix being an M -matrix.

Problem 5. Let Ω be a polygon in R^2 and consider the problem of finding a function $u(x, y)$ defined on Ω satisfying

$$(5.1) \quad -\Delta u + \underline{b} \cdot \nabla u = f, \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Here \underline{b} is a constant vector and f is a given smooth function.

- State a weak formulation of this problem and show that it is coercive on $H_0^1(\Omega)$. Let S_h be the finite element space of continuous piecewise linear finite elements over a triangulation \mathcal{T}_h of Ω and S_h^0 be the functions in S_h which vanish on $\partial\Omega$. Define the Galerkin finite element approximation $u_h \in S_h^0$.
- Derive an estimate for $\|u - \tilde{u}_h\|_{H^1(\Omega)}$ where \tilde{u}_h is the solution obtained when the right hand side in the finite element problem is replaced by the interpolant of f in S_h .
- Assuming that the solution of an appropriate adjoint problem has full elliptic regularity, derive an estimate for the error $\|u - \tilde{u}_h\|_{L^2(\Omega)}$.