NUMERICAL ANALYSIS QUALIFIER

May 2003

Do all of the following five problems.

Problem 1. Let A and B be $n \times n$ matrices and assume that B is symmetric and positive definite. Let $\|\cdot\|_B$ denote the matrix norm $\|v\|_B = (Bv, v)^{1/2}$ for all $v \in \mathbb{R}^n$ where $(\cdot,\cdot)$ denotes the dot product on $\mathbb{R}^n \times \mathbb{R}^n$. Assume that there are positive constants c_0 and c_1 satisfying

$$c_0 \|v\|_B^2 \le (Av, v) \quad \text{for all} \quad v \in \mathbb{R}^n \tag{1.1}$$

and

$$(Av, w) \le c_1 \|v\|_B \|w\|_B \quad \text{for all} \quad v, w \in \mathbb{R}^n. \tag{1.2}$$

(a) Show that $\|Av\|_{B^{-1}} \le c_1 \|v\|_B$ for all $v \in \mathbb{R}^n$.

(b) Consider the iterative method

$$x_{n+1} = x_n + \tau B^{-1}(f - Ax_n)$$

for solving $Ax = f$ and set $e_n = x - x_n$. Use Part (a) above to show that for $0 < \tau \le c_0/c_1^2$,

$$\|e_{n+1}\|_B \le (1 - c_0\tau)\|e_n\|_B.$$

Problem 2. Consider the Cauchy problem for $y(t)$:

$$y' = -\lambda(y - a(t)) + b(t), \; y(0) = a(0) = 0. \tag{2.1}$$

Here $\lambda > 0$ is a real parameter. It is easy to show that

$$\lim_{\lambda \to \infty} |y(t) - a(t)| = 0 \tag{2.2}$$

for any fixed t if a, b, and a' are continuous functions of t.

(a) Let U^n be the approximation of $y(t_n)$, $t_n = nh$ by the backward Euler method with step-size h. Show that this approximate solution shares the property (2.2) of the exact solution, namely

$$\lim_{\lambda \to \infty} |U^n - a(t_n)| = 0, \qquad \text{for any fixed } n \text{ and } h. \tag{2.3}$$

A discrete method that has the property is called ***L*-stable**.

(b) Consider the initial value problem

$$y' = f(y, t), \; y(t_0) = y_0$$

where $f(y, t)$ is a given function and y_0 is a given constant. Consider the general multistep method

$$a_k U_n + \cdots + a_0 U_{n-k} = h(b_k f_n + \cdots + b_0 f_{n-k}), \; U_0, \ldots, U_{k-1} \text{ given}.$$

Here U_i is an approximation to $y(t_i)$ and $f_i = f(U_i, t_i)$. Define what it means for this method to be A-stable. Give a necessary and sufficient condition for A-stability in terms of the behavior of the roots of certain polynomials. Is the Crank-Nicolson method A-stable? Justify your answer.

Problem 3. Consider the following two point boundary value problem:

$$-u''(x) + 5u'(x) = f(x),\ 0 \le x \le 1,\ -u'(0) + 5u(0) = 1,\ u(1) = 0.$$

(a) Derive a discretization of this problem on the mesh $x_j = jh$ for $j = 0, \dots, n$, $h = 1/n$ by using either linear finite elements or finite differences.
(b) Rewrite the scheme in the form $\mathcal{L}y_j = F_j,\ j = 1, \dots, n-1$ and $y_n = 0$, where

$$\mathcal{L}y_j \equiv \begin{cases} a_0 y_0 - c_0 y_1, & j = 0, \\ a_j y_j - b_j y_{j-1} - c_j y_{j+1}, & j = 1, \dots, n-1, \end{cases}$$

Using the maximum principle show that the resulting algebraic problem has a unique solution.
(c) Construct an appropriate barrier function and derive an *a priori* estimate for the discrete solution in maximum norm.

Problem 4. Let $L_x^2(0,1)$ be the weighted L^2-space with the norm

$$||f||_x^2 \equiv \int_0^1 x f^2(x) dx.$$

Let

$$V = \left\{ \phi :\ \text{locally integrable},\ \phi,\ \phi' \in L_x^2(0,1) \right\}.$$

V is a Hilbert space with norm $||\phi||_V^2 = ||\phi||_x^2 + ||\phi'||_x^2$.

(a) Show that if $\phi \in V$ then $|\phi(1)| \le ||\phi||_V$.
(b) Consider the boundary value problem: find $u(x)$ such that

$$\begin{aligned} -(xu')' + u &= f,\ x \in (0,1) \\ \lim_{x \to 0} (xu'(x)) &= 0, \\ u(1) &= 0. \end{aligned} \tag{4.1}$$

Derive the weak formulation of (4.1) in the space $V_0 = \{\phi \in V :\ \phi(1) = 0\}$. Show that the corresponding bilinear form is coercive and bounded in V-norm.
(c) Consider a uniform partition of $(0,1)$ into N equal subintervales and the corresponding finite element subspace of piecewise linear functions. Give an explicit representation of the entries of the global stiffness matrix corresponding to the nodal basis (you need not evaluate the integrals).

Problem 5. Let τ be a non-degenerate triangle in the plane with vertices $P_1,\ P_2,\ P_3$ and center of gravity $P_c = \frac{1}{3}(P_1 + P_2 + P_3)$. Consider the finite element $(\tau, \mathcal{P}_3, \Sigma)$, where $\mathcal{P}_3$ is the set of polynomials of degree at most three and $\Sigma = \{v(P_i), v_x(P_i), v_y(P_i),\ i = 1, 2, 3,\ v(P_c)\}$. This finite element is called the Hermite triangle.

(a) Show that Σ is $\mathcal{P}_3$-unisolvent set.
(b) Let $\mathcal{T}_h$ be a triangulation of a polygonal domain Ω and let V_h be the finite dimensional space (on Ω) associated with the finite element $(\tau, \mathcal{P}_3, \Sigma)$. Show that the functions in V_h are continuous.