**Numerical Analysis Qualifier,**
**May 27 , 2005, 1:00-5:00pm, Milner 216**
**Notes, books, and calculators are not authorized.**

**To pass you must provide satisfactory answers to problems 4 and 5 and at least to two problems among the three remaining problems.**

### Question 1

Let $I = [0,1]$. Let $k \geq 2$. Let $\{\xi_1, \ldots, \xi_{k-1}\}$ be the roots of $\mathcal{L}_k'$, where $\mathcal{L}_k$ is the Legendre polynomial of degree $k$. Recall that the Legendre polynomials are such that

$$\int_0^1 \mathcal{L}_m(t)\mathcal{L}_n(t)dt = \frac{1}{2m+1}\delta_{mn}, \quad 0 \leq m, n \leq k.$$

and $\mathbb{P}_n = \text{span}\{\mathcal{L}_0, \ldots, \mathcal{L}_n\}$.

(i) Show that $\int_0^1 \mathcal{L}_n(t)q(t)dt = 0$ for all $q \in \mathbb{P}_{n-1}$.

(ii) Let $\theta_0, \ldots, \theta_k$ be the Lagrange polynomials associated with the nodes $\{\xi_0, \xi_1, \ldots, \xi_{k-1}, \xi_k\}$ where $\xi_0 = 0$ and $\xi_k = 1$ (i.e., $\theta_i \in \mathbb{P}_k$ and $\theta_i(\xi_j) = \delta_{ij}$). How should the weights $\{\omega_0, \omega_1, \ldots, \omega_{k-1}, \omega_k\}$ be defined so that the quadrature formula

$$\int_0^1 f(t)dt \approx \sum_{i=0}^k \omega_i f(\xi_i)$$

is exact for the polynomials of degree at most $k$?

(iii) Show that, actually, the resulting quadrature is exact for all the polynomials of degree at most $2k-1$.

### Question 2

Consider the linear multistep method for $\frac{dy}{dt} = f(y,t)$

$$\sum_{j=0}^k a_{k-j} y_{n-j} = h \sum_{j=0}^k b_{k-j} f_{n-j}, \tag{1}$$

where we assume that $a_k = 1$ and $a_i \leq 0$ for $i = 0, \ldots, k-1$. Let $p(z) = z^k + a_{k-1}z^{k-1} + \ldots + a_0$.

(i) Assume that $p(1) = 0$. Show that the roots of $p$ are in the unit disk.

(ii) Show that it is not possible to have $p(1) = 0$, $p'(z) = 0$ and $|z| = 1$.

(iii) Show that if (1) is consistent, then (1) is stable.

(iv) Consider $k = 2$ and assume that $b_2 = 0$ (and $a_2 = 1$). Compute the coefficients $a_0$, $a_1$, $b_0$, and $b_1$ so that the method of the form of (1) is of order 3. Is it stable?

### Question 3

Let $A$ be a $n \times n$ real-valued symmetric positive definite matrix. Let $b \in \mathbb{R}^n$ and assume that $X \in \mathbb{R}^n$ solve $AX = b$. Let $\tau_1$, $\tau_2$ be two real numbers. The purpose of this problem is to analyze the following two-stage iterative algorithm:

$$X_{n+\frac{1}{2}} = X_n + \tau_1(b - AX_n),$$
$$X_{n+1} = X_{n+\frac{1}{2}} + \tau_2(b - AX_{n+\frac{1}{2}}).$$

(i) Let $e_i = X - X_i$ and $e_{i+\frac{1}{2}} = X - X_{i+\frac{1}{2}}$. Find the matrices $K_1$ and $K_2$ such that $e_{n+\frac{1}{2}} = K_1 e_n$ and $e_{n+1} = K_2 e_{n+\frac{1}{2}}$.

(ii) Find the matrix $K$ such that $e_{n+1} = Ke_n$.

(iii) If $\lambda$ is an eigenvalue of $A$, give the corresponding eigenvalue of $K$, say $\mu(\lambda)$.

(iv) Let $\lambda_m$ be the smallest eigenvalue of $A$ and let $\lambda_M$ be the largest. Make a rough graphic representation of the mapping $\lambda \longmapsto \mu(\lambda)$.

(v) Give a criterion for choosing $\tau_1$ and $\tau_2$ such that the above algorithm is the most rapidly convergent.

(vi) Let $\tilde{\lambda} = \frac{1}{2}(\lambda_M + \lambda_m)$ and $\hat{\lambda} = \frac{1}{2\sqrt{2}}(\lambda_M - \lambda_m)$. Choosing the above criterion for $\tau_1$ and $\tau_2$, express $\tau_1$, $\tau_2$, and the convergence ratio of the method in terms of $\tilde{\lambda}$ and $\hat{\lambda}$.

**Question 4**

Consider the boundary value problem:

$$-u'' + u = 0 \qquad x \in (0,1), \qquad u(0) = u(1) = 1. \tag{2}$$

(i) Introduce a weak formulation of this problem in appropriate Sobolev spaces of functions defined on the interval $(0,1)$.

(ii) Let $\mathcal{T}_h$ be the uniform partition of the interval $(0,1)$ into subintervals of size $h = 1/(N+1)$. Let $S_h$ be the space of the functions that are continuous on $[0,1]$, zero at 0 and 1, and piecewise linear on $\mathcal{T}_h$. Write the discrete counterpart to (2) in $S_h$. Denote by $u_h$ the corresponding approximate solution.

(iii) Let $x_i = ih$, $i = 1,\dots,N$ be the nodes of the mesh and let $\{\phi_1,\dots,\phi_N\}$ be the associated nodal basis of $S_h$. Using the nodal basis, compute the entries of the mass matrix $M$ associated with the term $u$ of (2). Compute the entries of the stiffness matrix $K$ associated with the term $u''$. Compute the entries of the global stiffness matrix $A = K + M$.

(iv) Show that the discrete problem in (ii) yields a linear system of the form $AU = hF$, where $U = (U_1, U_2, \dots, U_N)^T$ is the coordinate vector of $u_h$ relative to the nodal basis $\{\phi_1,\dots,\phi_N\}$. Give the entries of $F$.

(v) Let $I$ be the $N \times N$ identity matrix. Show that $M = hI + \alpha(h)K$ and find $\alpha(h)$.

(vi) Show that for all $1 \le i \le N$, $\min_{1\le j\le N}(F_j) \le U_i \le \max_{1\le j\le N}(F_j)$.

**Question 5**

Let $\Omega = ]0,1[$. Henceforth $L^1(\Omega)$ denotes the space of the scalar-valued functions that are integrable over $\Omega$. $W^{1,1}(\Omega)$ is the space of the scalar-valued functions in $L^1(\Omega)$ whose first weak derivatives are in $L^1(\Omega)$. We denote

$$\|v\|_{L^1} = \int_0^1 |v|, \qquad \|v\|_{W^{1,1}} = \|v\|_{L^1} + \|v'\|_{L^1}.$$

Let $f \in L^1(\Omega)$, and consider the following problem:

$$\begin{cases} \mu u + u_x = f, \\ u(0) = 0, \end{cases}$$

where $\mu$ is a nonnegative constant. Accept as a fact that for all $f \in L^1(\Omega) = V$ this problem has a unique solution in $W = \{w \in W^{1,1}(\Omega);\, w(0) = 0\}$.

Let $\mathcal{T}_h$ be a mesh of $\Omega$ composed of $N$ segments. Define the finite element spaces

$$W_h = \{w_h \in \mathcal{C}^0(\overline{\Omega});\, \forall K \in \mathcal{T}_h,\, w_{h|K} \in \mathbb{P}_1;\, w_h(0) = 0\},$$
$$V_h = \{v_h \in L^1(\Omega);\, \forall K \in \mathcal{T}_h,\, v_{h|K} \in \mathbb{P}_0\}.$$

The trial space $W_h$ is equipped with the norm of $W^{1,1}(\Omega)$ and the test space $V_h$ is equipped with the maximum norm: $\|v_h\|_{L^\infty} = \max_{K\in\mathcal{T}_h; x\in K} |v_h(x)|$. Introduce $a(u_h, v_h) := \int_0^1 (\mu u_h + u_{h,x})v_h$ and the following discrete problem:

$$\begin{cases} \text{Seek } u_h \in W_h \text{ such that} \\ a(u_h, v_h) = \int_0^1 f v_h, \quad \forall v_h \in V_h. \end{cases} \tag{3}$$

(i) Show that $a$ is bounded on $W_h \times V_h$.

(ii) For $w_h \in W_h$, let $\overline{w}_h \in V_h$ be the function such that the restriction of $\overline{w}_h$ to each mesh cell $K$ is the mean value of $w_h$ over this mesh cell, i.e., $\overline{w}_h|_K = \frac{1}{|K|}\int_K w_h$. Show that there is $c_1 > 0$, independent of $h$, such that

$$\|w_h - \overline{w}_h\|_{L^1} \le c_1 h \|w_h\|_{W^{1,1}}.$$

(iii) Denote by $\text{sign}(x)$ the sign function, i.e., $\text{sg}(x) = \frac{x}{|x|}$ if $x$ is not zero and $\text{sign}(0) = 0$. Let $w_h$ be a nonzero function in $W_h$. Set $z_h = \text{sign}(\mu\overline{w}_h + w_{h,x})$. Accept as a fact that $(z_h = 0) \Rightarrow (w_h = 0)$. Show that if $w_h \neq 0$ then

$$\frac{a(w_h, z_h)}{\|z_h\|_{L^\infty(\Omega)}} \ge \|\mu w_h + w_{h,x}\|_{L^1(\Omega)} - c_1\mu h\|w_h\|_{W^{1,1}(\Omega)}$$

(iv) Accept as a fact that there exists $\alpha > 0$ such that

$$\forall w \in W, \qquad \|\mu w + w_x\|_{L^1(\Omega)} \ge \alpha\|w\|_{W^{1,1}(\Omega)}.$$

Prove that there is $\gamma > 0$ and $h_0$ such that for all $h \le h_0$,

$$\inf_{w_h\in W_h} \sup_{v_h\in V_h} \frac{a(w_h, v_h)}{\|w_h\|_{W^{1,1}(\Omega)}\|v_h\|_{L^\infty(\Omega)}} \ge \gamma.$$

(v) Show that (3) has a unique solution.