NUMERICAL ANALYSIS QUALIFIER

May 23, 2006

Do all of the problems below. Make sure that you show your work in yes/no problems (simply answering yes or no will receive no credit).

Problem 1. (a) Let A be an $n \times n$ matrix and $\| \cdot \|_1$ denote the norm on $\mathbf{R}^n$ given by $\|v\|_1 = \sum_{i=1}^n |v_i|$. Show that

$$\|A\|_1 = \max_{j=1}^{n} \sum_{i=1}^{n} |A_{i,j}|.$$

(b) Let $\| \cdot \|_2$ denote the norm on $\mathbf{R}^n$ given by $\|v\|_2 = (\sum_{i=1}^n |v_i|^2)^{1/2}$. For

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},$$

compute $\|A\|_2$.

Problem 2. The following questions relate to (multistep) ODE schemes for solving

$$x'(t) = f(x(t), t).$$

(a) Show that the scheme

$$x_n - x_{n-1} = h(\theta f_n + (1-\theta) f_{n-1})$$

is A-stable for $1/2 \leq \theta \leq 1$.

(b) Consider the scheme

$$x_n - 3x_{n-1} + 2x_{n-2} = -\frac{1}{2} h(f_n + f_{n-1}).$$

Is it consistent? Is it stable?

(c) Consider the scheme

$$x_n - x_{n-2} = h(f_n - 3f_{n-1} + 4f_{n-2}).$$

Is it stable? Compute its order.

Problem 3. Consider the finite element space defined on triangles in $\mathbf{R}^2$ consisting of piecewise cubic functions $\mathcal{P}^3$. Consider as degrees of freedom:

(i) :The function values at the vertices.

(ii) :The values of the first derivatives at the vertices.

(iii) :The value of the function at the barycenter.

(a) Show that the degrees of freedom above form a unisolvent set. **Do this by using properties of the polynomials and not by writing down a huge system and trying to show that it is nonsingular.**

(b) State a theorem which provides a criterion that you can apply to determine when an assembled finite element space is H^1-conforming.
(c) Show that the assembled finite element space corresponding to this problem is H^1-conforming.
(d) Prove or disprove: The assembled finite element space corresponding to this problem is H^2-conforming.

Problem 4. Consider the one dimensional wave equation,

$$\begin{aligned} \zeta_{tt} - \zeta_{xx} &= 0, \quad \text{for} \quad (x,t) \in (0,1) \times (0,T], \\ \zeta(0,t) = \zeta(1,t) &= 0, \quad \text{for} \quad t \in (0,1], \\ \zeta(x,0) = \zeta_0(x), \quad \zeta_t(x,0) &= \eta_0(x), \quad \text{for} \quad x \in [0,1]. \end{aligned} \tag{4.1}$$

(a) Describe the (time continuous) semi-discrete approximation to (4.1) based on finite differences on a uniform grid in space consisting of m internal nodes.
(b) The semi-discrete method of Part (a) above can be written as a system of ODEs of the form

$$\begin{aligned} Z_{tt} + AZ &= 0, \quad \text{for} \quad t > 0 \\ Z(0) = Z_0, \quad Z_t(0) &= N_0. \end{aligned}$$

Here $Z(t), Z_0, N_0 \in \mathbf{R}^m$, A is a symmetric and positive definite $m \times m$ matrix, and Z_0 (resp. N_0) interpolates ζ_0 (resp. η_0). Let $N = Z_t$ then the above system leads to the first order system

$$N_t + AZ = 0, \qquad Z_t - N = 0. \tag{4.2}$$

Consider the following fully discrete scheme for (4.2) with step-size k:

$$\begin{aligned} \frac{Z_{n+1} - Z_n}{k} - N_n + \frac{k}{2} A Z_{n+1} &= 0, \\ \frac{N_{n+1} - N_n}{k} + A Z_{n+1} &= 0. \end{aligned} \tag{4.3}$$

Show that Z_k satisfies a three term recurrence by eliminating N.
(c) Show that (4.3) is unconditionally stable by using the three term relation of Part (b) above and expanding in terms of the eigenvectors of A.